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A SPHERICAL PROJECTION CHART FOR USE IN THE STUDY OF ELLIPTICALLY POLARIZED LIGHT

By F. E. Wright

[Geophysical Laboratory, Carnegie Institution of Washington, Washington, D. C., Received July 21, 1930]

ABSTRACT

The constants of an elliptical vibration are determined by transforming it either to a rectilinear vibration or to one of small, known ellipticity and then measuring the azimuth of the transformed vibration. From these measurements the constants of the elliptical vibration can be computed or ascertained by graphical methods. For this transformation, crystal plates of known thickness and birefringence are used. In each one of a series of transformations the incoming vibration is referred to the axes (α', γ') of the crystal plate along which the vibrations take place on passage through the plate; the phase lag or lead, introduced by the plate, is then added and the form of the vibration emerging from the plate thus ascertained. This procedure is repeated for each crystal plate and for the analyzer. Poincaré showed in 1892 how the relations of any given elliptical vibration can be represented by a point on a sphere or in a stereographic projection. His graphical method of representation is now in general use. In this connection a specially prepared angle or stereographic projection chart, as a base, greatly facilitates the graphical solutions. A brief description of this chart and of the methods developed for its use, as applied to the derivation of the equations and to the solution of problems in elliptical vibrations, is presented.

Throughout the paper the effect, on a given elliptical vibration represented by a point in the projection, of changing from one set of reference coordinate axes to a second set is obtained in the projection by a rotation about the vertical, $N-S$, axis through twice the angle between the sets. Similarly the effect of the introduction of a phase angle, ϕ, by a crystal plate on a given elliptical vibration, referred to the vibration axes (α', γ') of the crystal plate and represented by a point in the projection, is obtained by a rotation of the projection about the horizontal, $E-W$, axis through the angle, ϕ.

A stream of monochromatic light whose vibrations follow a definite path is said to be polarized. In general the vibration path is an ellipse; in special cases the ellipse may become a circle or a right line; these are the vibration paths of circularly and of plane polarized light respectively. The path of vibration is completely defined by the shape and position of the vibration ellipse and by the direction followed by the light vector along the ellipse. In actual practice the constants of the vibration

ellipse are ascertained by measuring the ratio of the principal axes of the ellipse (ellipticity $= \pm b/a$) and the angular position or azimuth, θ, of its major principal axis a with reference to some fixed line. To determine these constants the elliptical vibration is transformed either to a rectilinear vibration or to one of small, known ellipticity, approaching a rectilinear vibration, and then the azimuth of the transformed vibration is measured. From these data of measurement the constants of the elliptical vibration can be computed. In measurements of this kind the loss of light by reflection or absorption is negligible and is not taken into account in the theoretical treatment.

In the following paragraphs this subject of the analysis of elliptical vibrations is approached from the viewpoint of one accustomed to the use of spherical projection plots or nets; the equations are developed and treated with this in mind. Most of the results obtained are not novel. Poincare[1] suggested many years ago the use of the sphere and of a stereographic projection in this connection; but the method of approach and the graphical methods of solution outlined below seem simpler and more direct than methods heretofore employed and may merit brief description.

DERIVATION OF EQUATIONS

In a stream of monochromatic, elliptically polarized light let the components, u,v of the elliptical vibration, referred to the coordinate axes x,y be represented by the equations

$$u = A \cdot \cos \omega t \tag{1a}$$

$$v = B \cdot \cos (\omega t - \phi) \tag{1b}$$

in which A and B are the amplitudes of the accelerated and retarded components respectively; $\omega = 2\pi/T$ is the angular velocity of the vector; T, the vibration period; t is the elapsed time; ϕ, the angular difference in phase or phase lag of component v behind u. Both equations represent simple harmonic vibrations, as indicated in Fig. 1, in which x is the projection of the radius vector OA rotating with constant angular velocity ω; y is the projection of the radius vector OB rotating with the same angular speed ω, but lagging behind OA by the angular distance $BOC = \phi$.

[1] H. Poincaré, Théorie mathématique de la lumière. II, Chap. XII, 1892; see also J. Walker, Analytical theory of light, 303–318; 1904; F. Pockels, Lehrbuch der Kristalloptik, 11–13, 267–283; 1906; L. B. Tuckerman, Jr., Doubly refracting plates and elliptic analyzers. Univ. of Nebraska Studies *IX*, 157–219; 1909; C. A. Skinner, A universal polarimeter. J. Opt. Soc. & Rev. Sci. Inst. *10*, 490–520; 1925.

Eliminating t from equations 1a, 1b we obtain the equation of the ellipse that represents the vibration path of the vector whose components are u and v:

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} - \frac{2uv}{AB}\cos\phi = \sin^2\phi. \tag{2}$$

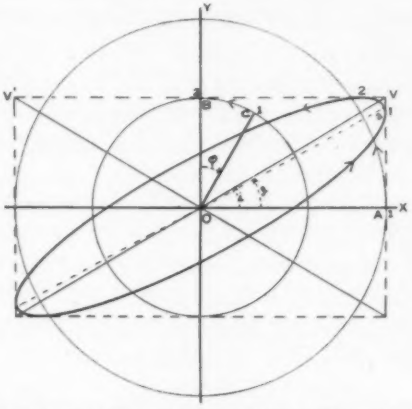


FIG. 1. *Elliptically polarized light may be resolved into two harmonic components of the same period but of different amplitudes and vibrating along the x and y axes, respectively; the y component lags behind the x component by the phase angle, ϕ. In the figure the two harmonic components are the x component of the vector OA and the y component of the vector OB, respectively, both vectors rotating at the same constant speed about O; the angular phase lag of OB behind OA is the angle BOC = ϕ.*

If the phase angle φ is 0, (or $2n\pi$); or π, (or $2(n-1)\pi$), equation 2 reduces to the equation

$$\frac{u}{A} - \frac{v}{B} = 0 \qquad \text{or} \qquad \frac{u}{A} + \frac{v}{B} = 0 \tag{3}$$

which represent the right lines OV and OV' of Fig. 1. These lines pass through the origin and include with the x-axis the position angles i and $\pi - i$, whose tangents are respectively

$$\tan i = \frac{B}{A} \quad \text{and} \quad \tan(\pi - i) = -\frac{B}{A}. \tag{4}$$

If the phase angle, ϕ, is $\pi/2$ or $(2n+1)\pi/2$ equation (2) becomes

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1 \tag{5}$$

the equation of an ellipse referred to its principal axis. The phase angle is 90° or any odd multiple thereof; the components u and v coincide with the principal axes a and b, of the ellipse. If in addition, $a=b$, the equation reduces to that of a circle and the resulting vibration path is circular.

To refer the equation of the ellipse of equation (2) to its principal axes, let the azimuth of the major axis a of the ellipse with respect to the

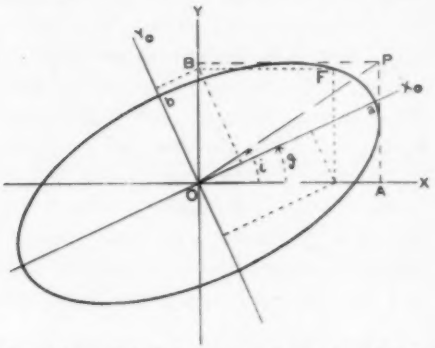


FIG. 2. *In this figure the angle included between the x-axis and the principal axis, a, of the ellipse is XOa=θ. The harmonic components of any point, F, on the ellipse along the OX and OY axes, and also along the OX_0 and OY_0 axes are indicated by the intercepts of the dotted lines on the several axes.*

x-axis be θ (angle $XO\ a$, Fig. 2). The components u_0, v_0 along the principal axes of the ellipse, a and b, are then

$$u_0 = u\cdot\cos\theta + v\cdot\sin\theta = a\cdot\cos(\omega t - \phi') \tag{6a}$$

$$v_0 = -u\cdot\sin\theta + v\cdot\cos\theta = b\cdot\cos(\omega t - \phi''). \tag{6b}$$

In these equations the angular phase lag of the v_0, (b), component over the u_0, (a), component is either

$$\phi'' - \phi' = (4n+1)\frac{\pi}{2} \qquad \text{or} \tag{7a}$$

$$\phi'' - \phi' = (4n-1)\frac{\pi}{2} \tag{7b}$$

which, for $n=0$, reduces to

$$\phi'' - \phi' = \frac{\pi}{2} \qquad \text{or} \tag{7c}$$

$$\phi'' - \phi' = -\frac{\pi}{2}. \tag{7d}$$

In Fig. 3a an ellipse with principal axes $a=0.8$, $b=0.4$, $b/a=0.5$, and $\phi=+\pi/2$ is shown; for the point (1) of the ellipse the rotating components of the vector have the positions marked (1); as the rotating vector components pass from (1) to (2) in a counter-clockwise direction the resultant vector of the ellipse passes from (1) to (2) also in a counter-clockwise direction. In Fig. 3b the phase lag is $-\pi/2$ with the same principal axes; on the ellipse the point (1) corresponds to the points (1) of the x and y harmonic components. As these rotating vector components pass from (1) to (2) in a counter-clockwise direction the resultant ellipse vector passes from (1) to (2) in a clockwise direction. In the first case the character of the elliptical vibration is laevogyre or positive; in the second, dextrogyre or negative. In the equations to be deduced, the direction of movement of the vector of the elliptical vibration is indicated by the sign of the principal axes. Let $\tan\epsilon=b/a$; then $\tan\epsilon=+b/a$ refers to counter-clockwise or laevogyre rotation of the vector and $\tan\epsilon=-b/a$, to dextrogyre or clockwise movement.*

On substituting from equations 1a, 1b the values of u and v in 6a, 6b; then equating the coefficients of $\cos\omega t$ and $\sin\omega t$ and having regard for the relations of 7a, 7b, we obtain the equations

Character of elliptical vibration (+ *or* −)

Equations	Phase angle ϕ			
	0 to $\frac{\pi}{2}$	$\frac{\pi}{2}$ to π	π to $\frac{3}{2}\pi$	$\frac{3}{2}\pi$ to 2π
$x=A\cdot\cos\omega t$ $y=B\cdot\cos(\omega t-\phi)$	+	+	−	−
$x=A\cdot\cos(\omega t+\phi)$ $y=B\cdot\cos\omega t$	+	+	−	−
$x=A\cdot\sin\omega t$ $y=B\cdot\sin(\omega t-\phi)$	+	+	−	−
$x=A\cdot\sin(\omega t+\phi)$ $y=B\cdot\sin\omega t$	+	+	−	−

If the sine function, $x=A\cdot\sin\omega t$, is used, the rotating u radius vector (harmonic component x-axis) is at right angles to the x-axis for $t=0$; if the cosine function, $x=A\cdot\cos\omega t$, is used, the rotating u radius vector coincides with the x-axis for $t=0$.

* The character and direction of movement of the vector whether clockwise (negative −) or counter-clockwise (positive +) depends on the phase angle ϕ. The harmonic components of the vector may be expressed either as sine or as cosine functions as follows:

$$A \cdot \cos\theta + B \cdot \sin\theta \cdot \cos\phi = a \cdot \cos\phi \tag{8a}$$

$$B \cdot \sin\theta \cdot \sin\phi = a \cdot \sin\phi' \tag{8b}$$

$$-A \cdot \sin\theta + B \cdot \cos\theta \cdot \cos\phi = b \cdot \cos\phi'' = \mp b \cdot \sin\phi' \tag{8c}$$

$$B \cdot \cos\theta \cdot \sin\phi = b \cdot \sin\phi'' = \pm b \cdot \cos\phi' \tag{8d}$$

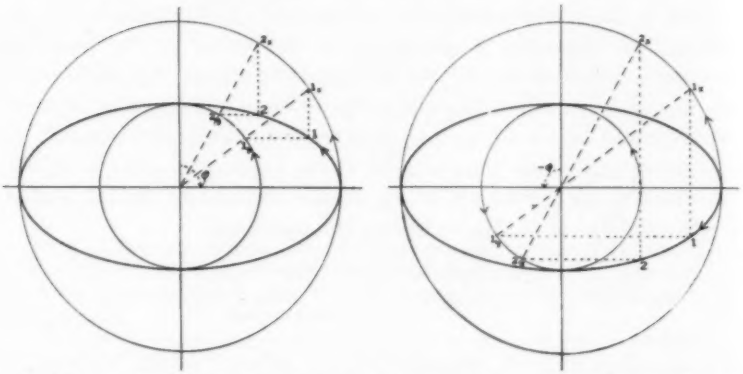
and deduce from them

$$a^2 + b^2 = A^2 + B^2 \tag{9a}$$

$$a^2 - b^2 = (A^2 - B^2) \cdot \cos 2\theta + 2AB \cdot \sin 2\theta \cdot \cos\phi \tag{9b}$$

$$\pm ab = AB \cdot \sin\phi \tag{9c}$$

$$\tan 2\theta = \frac{2AB}{A^2 - B^2} \cdot \cos\phi. \tag{10}$$



FIGS. 3a, 3b *illustrate the dependence of the direction of movement of the elliptic vector, whether counter-clockwise or clockwise, on the phase difference between the two harmonic x and y components.*

To simplify these expressions, let the two auxiliary angles ϵ and i be introduced. By definition

$$\tan i = \frac{B}{A}, \text{ or } \sin 2i = \frac{2AB}{A^2 + B^2}, \cos 2i = \frac{A^2 - B^2}{A^2 + B^2}, \tan 2i = \frac{2AB}{A^2 - B^2} \tag{11}$$

$$\tan\epsilon = \frac{b}{a}, \text{ or } \sin 2\epsilon = \frac{2ab}{a^2 + b^2}, \cos 2\epsilon = \frac{a^2 - b^2}{a^2 + b^2}, \tan 2\epsilon = \frac{2ab}{a^2 - b^2}. \tag{12}$$

Substituting these angles in the expressions 8 to 10 we derive the equations

$$\cos 2\epsilon = \cos 2i \cdot \cos 2\theta + \sin 2i \cdot \sin 2\theta \cdot \cos\phi \tag{13}$$

$$\pm \sin 2\epsilon = \sin 2i \cdot \sin\phi \tag{14}$$

$$\tan 2\theta = \tan 2i \cdot \cos\phi \tag{15}$$

and from (13) to (15) in turn

$$\cos 2i = \cos 2\epsilon \cdot \cos 2\theta \tag{16}$$

$$\pm \tan 2\epsilon = \sin 2\theta \cdot \tan \phi. \tag{17}$$

These equations, 14 to 17, are familiar equations in spherical projection work and suffice to indicate a method for the graphical solution of problems in the analysis of elliptical vibrations.

On the sphere (Fig. 4) let E represent the direction of the x-axis; the direction of the major axis, a, of the ellipse is indicated by D which is distant 2θ from E; the angular distance between C and D along the

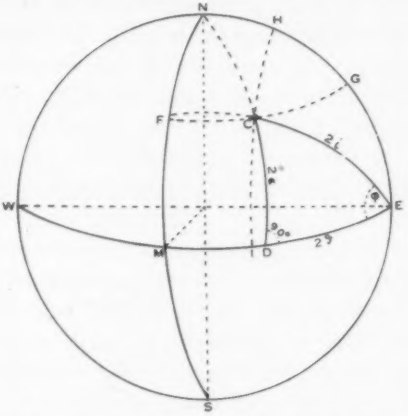


FIG. 4 *illustrates the Poincaré construction on the sphere for the representation of elliptically polarized light. From the spherical triangle CDE, the relations given in equations 13 to 18 can be read off directly.*

latitude arc NCD is 2ϵ; the angular distance, C to E, $2i$; the angle $EDC = 90°$, while angle $DEC = \phi$, the angular phase lag of the component v behind u. From the right angle, spherical triangle CED the relations expressed in equations 14 to 17 can be read off directly.

If the axial ratio or ellipticity of the vibration ellipse, $\tan \epsilon = b/a$, is maintained constant and the ellipse is referred to different coordinate axes, the point C on the sphere maintains its angular distance from the equator constant and the latitude arc CD remains unchanged in length; but the angular distance $CE = 2i$ changes and with it the angle $DE = 2\theta$ and the phase angle $\phi = DEC$. As the sphere is rotated about the N-S axis, while E is held fixed in direction in space, the point C moves along the parallel of latitude, CG; the angle $ED = 2\theta$ varies directly

with the angular rotation; the angle $\phi = DEC$ can be measured directly on the sphere.—For a point C located on the equator, $2\epsilon = 0$, also $\phi = 0$, the vibration is rectilinear, and the plane of vibration includes the angle i with the x-axis.—If C is located on the meridian NES the phase angle $\phi = \pm\pi/2$; the azimuth angle $\theta = 0$, in other words, the coordinate axes coincide with the principal ellipse axes; $2\epsilon = 2i$.—Both the N and S poles represent circular vibrations, $2\epsilon = 90°$, $\epsilon = 45°$, $\tan\epsilon = \pm b/a = \pm 1$, and $\phi = \pm 90°$. For the N-pole the light vector appears to the observer looking toward the light source to travel counter-clockwise in its orbit. This is true for all points on the Northern hemisphere; all points on the Southern hemisphere represent clockwise vibrations.

If the coordinate axes of reference, x, y, are maintained fixed in in position the angle $2i$ which defines the amplitude ratio ($\tan i = B/A$) is constant; but the phase angle ϕ and with it the angles 2θ and 2ϵ are changed. This means a rotation of the sphere about the E-W axis. During the rotation the point C maintains its distance from the pole E constant and travels along the small vertical circle ICH (Fig.4) parallel to the meridian NMS.

Rotation of the sphere, therefore, about the vertical axis is equivalent to holding the ellipticity $\tan\epsilon = \pm b/a$ constant, or to referring a given ellipse to different sets of coordinate axes. Rotation of the sphere about the horizontal axis EW is equivalent to angle ϕ, but changing the phase holding the amplitude ratio, $\tan i = B/A$, constant.—If the angle $2i$ is held constant, equations 9a and 11 show that A^2 and B^2 are then each constant; A and B are the amplitudes of the components of the beam. The intensity of illumination by the beam, as seen through the analyzer set at the angle i with the x-axis is measured by A^2. Therefore all points on a small vertical circle concentric about E and distant $2i$ from it represent vibrations that give the same brightness to the field when viewed through the analyzer.

If, as in a crystal plate of given thickness and birefringence, the phase angle, ϕ, remains constant for a given wave length, the position of the polarizer (angle i) may be changed and with it the azimuth angle θ, and the ellipticity $\tan\epsilon$, of the emergent beam. On the sphere this change is equivalent to a shift of the point C along the great circle EC which includes the angle ϕ with the equator.

Projection on a sphere is not convenient. In its place a projection plot or net (angle or stereographic projection) may be used on which

both meridians and parallels of latitude are drawn for north and south poles and for east and west poles (Fig. 5).* In this plot, which is re-

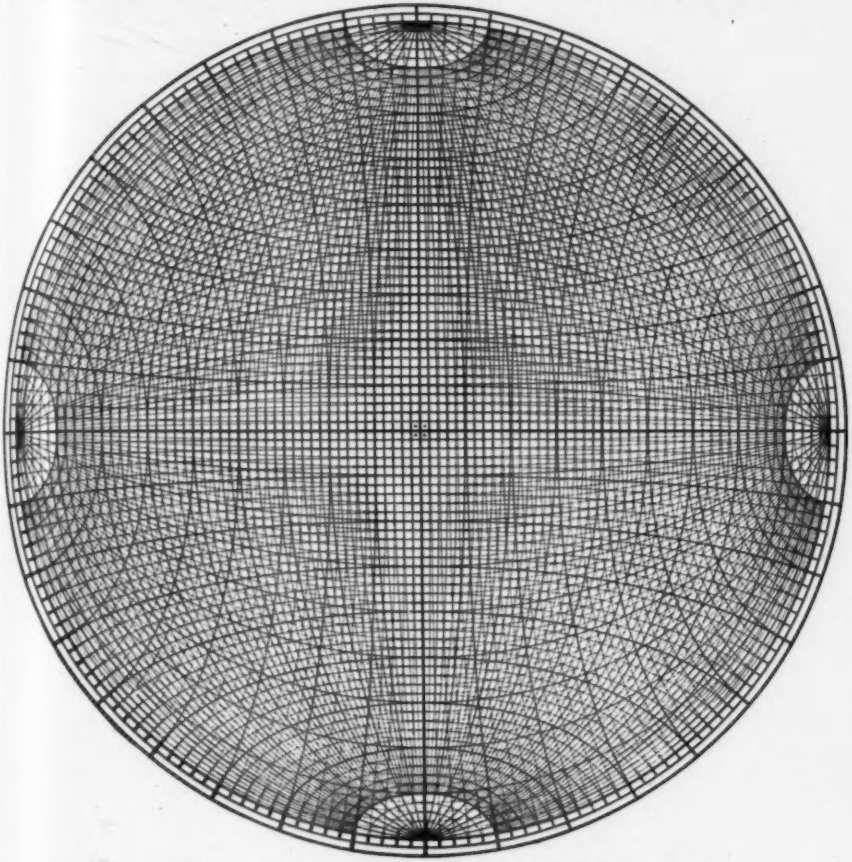


FIG. 5. *Angle projection chart for use in the study of elliptically polarized light. The great and also the small circles are 2° apart.*

* The base for this angle-projection chart first appeared in Publication 158 of the Carnegie Institution of Washington. This was copied by photography on a reduced scale and then printed and overprinted at 90° at the U. S. Geological Survey. For his own work the writer has a large *angle* projection chart 40 cm in diameter and with curves drawn 1° apart; also a similarly doubly printed *stereographic* projection chart, 40 cm in diameter and with curves drawn 1° apart. The base for this second projection chart was the stereographic projection chart published in 1888 by the Hydrographic Office of the U. S. Navy. The writer is indebted both to the U. S. Geological Survey and to the U. S. Hydrographic Office for aid in connection with the reproduction of these charts.

produced on a greatly reduced scale the curves are drawn 2° apart. It serves as a base on which a sheet of tracing paper is placed; the required rotations about the *N-S* and *E-W* axes are performed without disturbing the paper. Angular measurements in the projection are made with the aid of a sheet of tracing paper placed directly over the projection net. This method was used in America in the fifties of the last century by Chauvenet and others in the solution of navigation problems. Thirty years later Sigsbee used it for the same purpose. Still later Penfield in America and Wulff in Russia adopted it in crystallographical and petrological work.

The use of this projection plot or net is best illustrated by the graphical solution of several of the problems that may arise.

(1) Let the components of a given elliptical vibration be

$$u = A \cdot \cos \omega t \tag{1a}$$

$$v = B \cdot \cos (\omega t - \phi) \tag{1b}$$

when referred to an x, y set of coordinates. How does the amplitude ratio $B/A = \tan i$, also the phase angle ϕ, change when the given

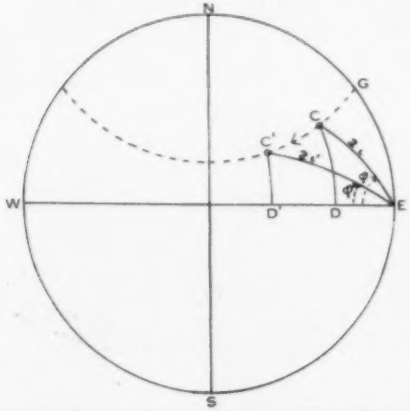


FIG. 6 *indicates the change in phase angle* ϕ *and in the amplitude ratio tan* $i = B/A$, *when the elliptical vibration represented by C is referred to another set of coordinate axes which include an angle* θ' *with the original set.*

elliptical vibration is referred to another set of rectangular coordinates which include an angle θ' with the first set? The angles $2i$ and ϕ, given by the equations 1a, 1b, suffice to locate the point C in the projection and with it the right angle triangle ECD (Fig. 6). The angle ED in this triangle is 2θ, or twice the angle included between the x-axis and

the major axis, a, of the vibration ellipse. If the angle θ be increased by θ', the angle between the x'-axis and the a-ellipse axis is $\theta+\theta'$. In the projection this is equivalent to a rotation about the vertical, N-S, axis through an angle $2\theta'$; C is shifted to C' and D to D'; $C'D'=CD$; $ED'=2(\theta+\theta')$. From the triangle $C'D'E$ the angles $2i'$ and ϕ' can be read off directly. The new amplitude ratio is $\tan i'=B'/A'$; the new phase lag angle is $D'E'C=\phi'$; the ellipticity, $\tan \epsilon=\pm b/a$, remains unchanged.

(2) What is the effect, in equations 1a, 1b, of changing the phase lag from ϕ to ϕ' but maintaining the amplitude ratio $\tan i=B/A$ constant? In the projection this means that $CE=2i$ remains constant while angle $DEC=\phi$ is increased to $D'EC'=\phi'$ (Fig. 7); the projection is

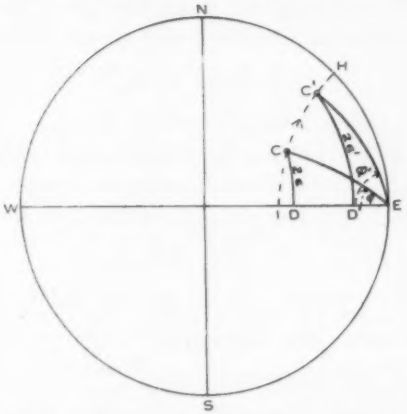


FIG. 7. *On changing the phase lag from ϕ to ϕ', but maintaining the amplitude ratio constant, the point C in the projection is shifted to C'; the ellipticity is changed from $\tan \epsilon=b/a$ to $\tan \epsilon'=b'/a'$ and the azimuth angle, from θ to θ'.*

rotated about the horizontal axis E-W through the angle $CEC'=\phi'-\phi$. The point C travels along the small vertical circle ICC' to C'; in the triangle $C'ED'$, $C'D'=2\epsilon'$ and $D'E=2\theta'$. As the angle ϕ increases the point C' of the triangle rises to higher latitudes, the angle 2ϵ increases and 2θ decreases.

(3) Deduce the effects produced on plane polarized light on transmission through a birefracting crystal plate. The movement of the light vector emerging from the polarizer and entering the crystal plate perpendicularly is given by

$$K \cdot \cos \omega t$$

in which K is the initial amplitude. Let $K=1$; this means that all

amplitudes are to be measured in terms of K. Let the plane of vibration of the entering wave include the angle i with the vibration direction of the faster wave, α', of the crystal plate. The components u and v of the light vector on entering the crystal plate along its two vibration directions, α' and γ', are then

$$u = \cos i \cdot \cos \omega t$$

$$v = \sin i \cdot \cos \omega t.$$

The light wave, vibrating along α' travels faster than that vibrating along γ'; on emergence from the plate, the light wave u is ahead of v a certain distance, Δ, called the path difference. If d is the thickness of the plate in millimeters and α', γ' are the two refractive indices, the path difference increases with d and with $\gamma' - \alpha'$ and is equal to

$$\Delta = d(\gamma' - \alpha').$$

The number of wave lengths by which the wave α' is ahead of the wave γ' is found by dividing Δ by the wave length λ. In harmonic motion a wave length represents a complete cycle or rotation of the radius vector through 360° or 2π. The phase angle is therefore

$$\phi = \frac{2\pi\Delta}{\lambda}$$

expressed in radians. The components of the resulting vibration are

$$u = \cos i \cdot \cos \omega t \tag{18a}$$

$$v = \sin i \cdot \cos (\omega t - \phi). \tag{18b}$$

The angle, ϕ, is the angular lag in phase of v behind u; the time t is counted from the instant when the radius vector of the u-component coincides in direction with the plane of vibration of the polarizer. From equations 18a, 18b we find for

$\omega t = 0,$	$u = \cos i,$	$v = \sin i \cdot \cos \phi$
$\omega t = \frac{\pi}{2},$	$u = 0$	$v = \sin i \cdot \sin \phi$
$\omega t = \phi$	$u = \cos i \cdot \cos \phi$	$v = \sin i$
$\omega t = \phi + \frac{\pi}{2}$	$u = -\cos i \cdot \sin \phi$	$v = 0.$

Equations 18a, 18b are identical in form with equations 1a, 1b, if $\cos i = M \cdot A$ and $\sin i = M \cdot B$, M being a proportionality constant;

this factor disappears in the ratio B/A. Therefore equations 13 to 17 are directly applicable and the resulting vibration can be represented by a point C in the projection.

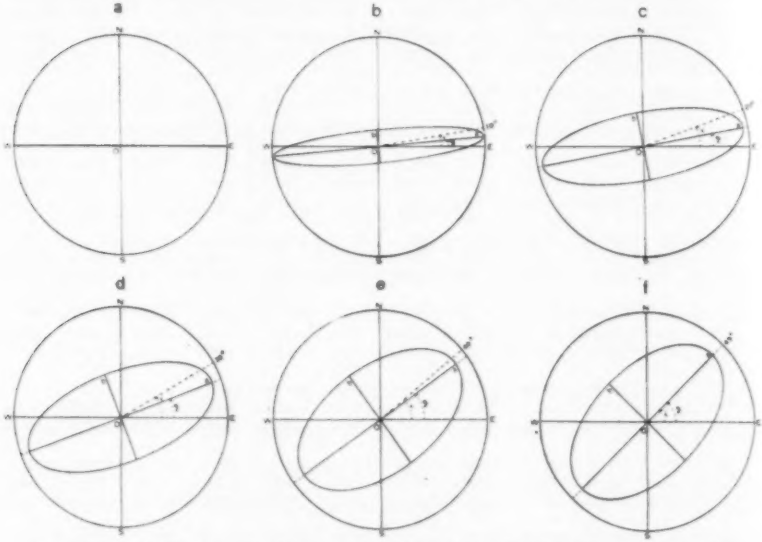
TABLE 1.

	$\phi=0°$	$\phi=10°$	$\phi=20°$	$\phi=30°$	$\phi=40°$	$\phi=45°$	$\phi=50°$	$\phi=60°$	$\phi=70°$	$\phi=80°$	$\phi=90°$
i	θ a b	θ a b	θ a b	θ a b	θ a b	θ a b	θ a b	θ a b	θ a b	θ a b	θ a b
0°	0° 1.000 .000	0° 1.000 .0000	0° 1.000 .0000	0° 1.000 .0000	0° 1.000 .0000	0° 1.000 .0000	0° 1.000 .0000	0° 1.000 .0000	0° 1.000 .0000	0° 1.000 .0000	0° 1.000 .0000
10°	10° 1.000 .000	9° 52′ .99956 .02967	9° 26′ .99828 .05759	8° 45′ .99631 .08585	7° 48′ .99386 .11060	7° 13′ .99255 .12187	6° 35′ .99123 .13111	5° 10′ .98871 .14982	3° 33′ .98664 .16290	1° 49′ .98528 .17093	0° .98481 .17365
20°	20° 1.000 .000	19° 47′ .99828 .05759	19° 08′ .99327 .11580	18° 00′ .98533 .17064	16° 22′ .97511 .22169	15° 21′ .96945 .24531	14° 10′ .96359 .26738	11° 23′ .95204 .30597	8° 01′ .94200 .33549	4° 09′ .93514 .35429	0° .93969 .34202
30°	30° 1.000 .000	29° 49′ .99715 .07541	29° 13′ .98871 .14982	28° 09′ .97502 .22212	26° 30′ .95673 .29098	25° 23′ .94618 .32364	24° 02′ .93493 .35484	20° 27′ .91140 .41151	15° 20′ .88915 .45762	8° 22′ .87235 .48888	0° .86603 .50000
40°	40° 1.000 .000	39° 56′ .99644 .08440	39° 41′ .98528 .17093	39° 15′ .96705 .25460	38° 31′ .94186 .33600	38° 00′ .92675 .37569	37° 20′ .91008 .41443	35° 17′ .87242 .48875	31° 22′ .83034 .55726	22° 17′ .78855 .61497	0° .76604 .64279
45°	45° 1.000 .000	45° 00′ .99619 .08716	45° 00′ .98481 .17365	45° 00′ .96593 .25882	45° 00′ .93969 .34202	45° 00′ .92388 .38268	45° 00′ .90631 .42262	45° 00′ .86603 .50000	45° 00′ .81915 .57358	45° 00′ .76604 .64279	—— .70711 .70711
50°	50° 1.000 .000	50° 05′ .99644 .08440	50° 19′ .98528 .17093	50° 45′ .96705 .25460	51° 29′ .94186 .33600	52° 00′ .92675 .37569	52° 40′ .91008 .41443	54° 43′ .87242 .48875	58° 38′ .83034 .55726	67° 43′ .78855 .61497	90° .76604 .64279
60°	60° 1.000 .000	60° 12′ .99715 .07541	60° 47′ .98871 .14982	61° 51′ .97502 .22212	63° 30′ .95673 .29098	64° 37′ .94618 .32364	65° 58′ .93493 .35484	69° 33′ .91140 .41151	74° 41′ .88915 .45762	81° 38′ .87235 .48888	90° .86603 .50000
70°	70° 1.000 .000	70° 13′ .99828 .05759	70° 53′ .99327 .11580	72° 00′ .98533 .17064	73° 38′ .97511 .22169	74° 40′ .96945 .24531	75° 50′ .96359 .26738	78° 37′ .95204 .30597	82° 00′ .94200 .33549	85° 51′ .93514 .35429	90° .93969 .34202
80°	80° 1.000 .000	80° 09′ .99956 .02967	80° 34′ .99828 .05759	81° 15′ .99631 .08585	82° 12′ .99386 .11060	82° 47′ .99255 .12187	83° 25′ .99123 .13111	84° 51′ .98871 .14982	86° 27′ .98664 .16290	88° 12′ .98528 .17093	90° .98481 .17365
90°	90° 1.000 .000	90° 1.000 .0000	90° 1.000 .0000	90° 1.000 .0000	90° 1.000 .0000	90° 1.000 .0000	90° 1.000 .0000	90° 1.000 .0000	90° 1.000 .0000	90° 1.000 .0000	90° 1.000 .0000

If the crystal plate be rotated, the angle i changes, but not the phase angle, ϕ, which is a function solely of the thickness and birefringence of the plate. If in the projection the angle ϕ is held constant and the angle $2i$ is allowed to vary, the point C travels along the great circle EC' and

with its change in position the arcs $C'D' = 2\epsilon'$ and $ED' = 2\theta'$ change. The ellipticity for any given angle of rotation, i', of the plate is $\tan \epsilon' = \pm b/a$.

To show how θ, a, and b change with the position angle i and with the phase angle ϕ, a series of values is listed in Table 1. The shapes and positions of the vibration ellipses of light emerging from a crystal plate whose phase angle for a given wave length is $\phi = 60°$ and whose vibration direction α' includes the angle $i = 0°$, 10°, 20°, 30°, 40°, 45°, with the incident plane polarized light are shown in Figs. 8a to 8f. In these



FIGS. 8a to 8f *illustrate the shapes and positions of the vibration ellipse emerging from a crystal plate of phase angle* $\phi = 60°$*; its vibration direction,* α'*, includes the angle* $i = 0°$, 10°, 20°, 30°, 40°, *and* 45°, *respectively, with the incident plane polarized light.*

figures the position angles, i, are marked along the outer edge of the circle; the azimuth angles, θ, are the angles included between the x-axis (α' direction) and the major axis, a, of the resulting elliptical vibration. For $i = 0°$, 45° and 90°, the azimuth angle θ is identical with the position angle i; for all other angles in the first quadrant it is not the same; similar relations obtain in the remaining quadrants. The change in azimuth angle θ with change in position angle, i, is shown, for different phase angles, ϕ, by the curves in Fig. 9. For small phase angles, less than 30°, i and θ differ but little; for larger phase angles, ϕ, the azimuth angle θ is noticeably less than i. For $\phi = 90°$, θ is zero for all values of

up to 45°; at this angle its value passes abruptly to 90° and remains thereafter constant to $i = 90°$; similar relations obtain for the remaining quadrants.

(4) Deduce the effects produced on plane polarized light on trans-

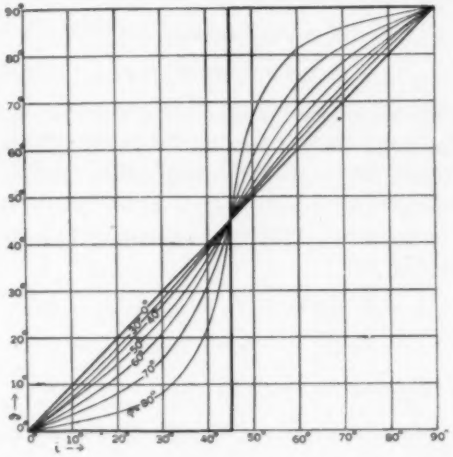


FIG. 9. *The curves in this figure indicate the change in azimuth angle, θ (ordinates), with change in position angle, i (abscissae), for different phase angles, ϕ (curves).*

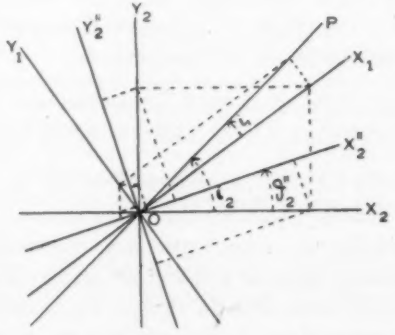


FIG. 10. *Diagram to illustrate the positions of the axes of the two superimposed crystal plates. PO represents the direction of the vibration of the plane polarized light entering the first crystal plate with axes X_1O and Y_1O; the position angle $X_1OP = i_1$. After emerging from this plate the light enters the second crystal plate with axes, X_2O and Y_2O; the position angle $X_2OP = i_2$. The azimuth angle $X_2OX_2'' = \theta_2''$ is the angle included between the major axis a_2 of the emergent elliptical vibration with the X_2O axis. The dotted lines indicate at their intersections with the several axes the vector components of the successive elliptical vibrations.*

mission through two superimposed birefracting crystal plates. Let the subscripts 1 and 2 be used to represent the several elements that characterize the first and second plates respectively (Fig. 10). The components of the movement of the light vector, on emerging from the first plate, are then, as indicated in the foregoing section,

$$u_1 = \cos i \cdot \cos \omega t \tag{18a}$$

$$v_1 = \sin i \cdot \cos (\omega t - \phi_1). \tag{18b}$$

On entering the second birefracting plate the vibrations of the light wave take place along the α'_2 and γ'_2 directions; during the passage through the plate the wave vibrating along the γ_2' direction is retarded; on emergence its phase lag is ϕ_2 behind the wave vibrating along α'_2. The components of the vector emerging from the second plate are accordingly (Fig. 10):

$$u_2 = u_1 \cdot \cos (i_2 - i_1) - v_1 \cdot \sin (i_2 - i_1) \tag{19a}$$

$$v_2 = u_1 \cdot \sin (i_2 - i_1) + v_1 \cdot \cos (i_2 - i_1) \tag{19b}$$

or with the aid of equations 18a, 18b

$$u_2 = \cos i_1 \cdot \cos (i_2 - i_1) \cdot \cos \omega t - \sin i_1 \cdot \sin (i_2 - i_1) \cdot \cos (\omega t - \phi_1) \tag{19c}$$

$$v_2 = \cos i_1 \cdot \sin (i_2 - i_1) \cdot \cos (\omega t - \phi_2) + \sin i_1 \cdot \cos (i_2 - i_1) \cos(\omega t - \phi_1 - \phi_2). \tag{19d}$$

Let θ''_2 be the angle included between the x_2-axis and the major axis, a_2, of the vibration ellipse inscribed by the vector whose components are u_2 and v_2. Referring the ellipse to its principal axes as coordinate axes we have, as in equations 6a, 6b, the components

$$u_0'' = u_2 \cdot \cos \theta_2'' + v_2 \cdot \sin \theta_2'' = a_2 \cdot \cos (\omega t - \phi') \tag{20a}$$

$$v_0'' = - u_2 \cdot \sin \theta_2'' + v_2 \cdot \cos \theta_2'' = b_2 \cdot \cos (\omega t - \phi'') \tag{20b}$$

in which $\phi'' - \phi' = (4n+1) \cdot \pi/2$, as in equations 7a, 7b.

On substituting in these equations the values u_2 and v_2 from equations 19c, 19d, and expanding the cosine expressions containing ωt, and then equating the coefficients of cos ωt and and sin ωt, we obtain a set of four equations analogous to equations 8a to 8d. From these equations, in turn, the following expressions can be derived, if we introduce the angles $\tan \epsilon_1 = \pm b_1/a_1$ and $\tan \epsilon_2'' = \pm b''_2/a''_2$ for the ellipticities;

$$\pm \sin 2\epsilon_2'' = \cos 2\epsilon_1 \cdot \sin 2(\theta_1 + i_2 - i_1) \cdot \sin \phi_2 \pm \sin 2\epsilon_1 \cdot \cos \phi_2 \tag{21}$$

$$\tan 2\theta_2'' = \frac{\sin 2(\theta_1 + i_2 - i_1) \cdot \cos \phi_2 \mp \tan 2\epsilon_1 \cdot \sin \phi_2}{\cos 2(\theta_1 + i_2 - i_1)}. \tag{22}$$

Were the second plate to introduce no phase difference or $\phi_2 = 0$; in other words, were the elliptical vibration after emergence from the first plate to be referred to the x_2, y_2 coordinate axes, equations 21 and 22 reduce to

$$\epsilon_2' = \epsilon_1 \tag{23a}$$

$$\theta_2' = \theta_1 + i_2 - i_1, \text{ or } \theta_2' - \theta_1 = i_2 - i_1. \tag{23b}$$

In the projection, equations 23a, 23b signify a rotation about the N-S axis through the angle, $i_2 - i_1$, thus referring the elliptical vibration from the first plate alone to the x_2, y_2 coordinate system. By this rotation (Fig. 11) the point C_1 passes to C_2' along the small circle C_1C_2'; the angle

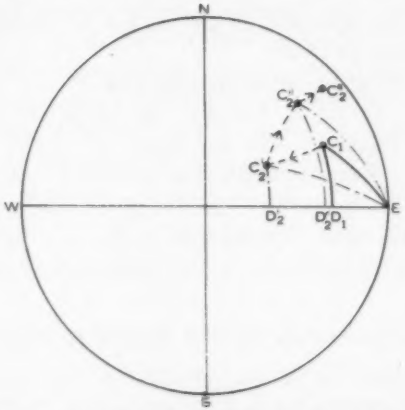


FIG. 11. *In this figure C_1 represents the elliptical vibration emerging from the first crystal plate. This vibration is referred to the vibration axes of the second plate by rotation of the projection about the vertical, $N-S$, axis through the angle $2(i_2-i_1)$ and C_1 is shifted to C_2'. The phase angle ϕ_2 is introduced by rotation of the projection about the horizontal, $E-W$, axis through ϕ_2; whereby C_2' is shifted to C_2''. To refer the vibration ellipse represented by C_2'' to the original axes, x_1, y_1, the projection is rotated about the vertical axis through the angle $2(i_1-i_2)$ and C_2'' is shifted to C_2'''.*

$C_2'D_2' = C_1D_1 = 2\epsilon_1$. The angles $C_2'E = 2i_2'$ and $C_2'ED_2' = \phi_2'$ can either be read off directly in the projection plot or computed by aid of the equations

$$\cos 2i_2' = \cos 2\epsilon_1 \cdot \cos 2(\theta_1 + i_2 - i_1) \tag{24}$$

$$\cot \phi_2' = \cot 2\epsilon_1 \cdot \sin 2(\theta_1 + i_2 - i_1) \tag{25}$$

$$\cot 2i_2' = \cos \phi_2 \cdot \cot 2(\theta_1 + i_2 - i_1). \tag{26}$$

The next step is to rotate the projection about the E-W axis through the angle, ϕ_2; the point C_2' moves to C_2'' (Fig. 11) along the small

vertical circle $C_2'C_2''$. For the triangle $EC_2''D_2''$ equations 21 and 22 are valid; but the angles can be read off directly on the projection net. The point C_2'' represents the elliptical vibration of the light emerging from the second crystal plate; the azimuth of its principal axis a_2 is half the angle $ED_2''=2\theta_2''$; its phase angle is $C_2''ED_2''=\phi_2'+\phi_2=\phi_2''$; also $C_2''D_2''=2\epsilon_2''$ and $C_2''E=C_2'E=2i_2'=2i_2''$. These angles refer to the x_2, y_2 coordinates. In this right angle spherical triangle $C_2''D_2''E$ we read off directly the equations

$$\sin 2\epsilon_2'' = \sin 2i_2' \cdot \sin(\phi_2' + \phi_2) \tag{27}$$

$$\tan 2\theta_2'' = \tan 2i_2' \cdot \cos(\phi_2' + \phi_2) \tag{28}$$

from which, in turn, equations 21 and 22 can be readily deduced with the aid of equations 24 to 26.

To refer the vibration ellipse represented by C_2'' to the original coordinate axes x_1, y_1,(α_1', γ_1'), rotate the projection about the N-S axis through the angle $2(i_1-i_2)$ and shift C_2'' to C_2''' (Fig. 11); the ellipticity of the vibration ellipse $\tan \epsilon_2'' = \pm b_2''/a_2''$ remains unchanged during this operation; but the phase angle ϕ_2'' changes to ϕ_2'''; the azimuth angle becomes $\theta_2'''=(\theta_2''+i_1-i_2)$; these angles can either be read off directly in the projection or computed from the equations

$$\cos 2i_2''' = \cos 2\epsilon_2'' \cdot \cos 2(\theta_2'' + i_1 - i_2) \tag{29}$$

$$\cot \phi_2''' = \cot 2\epsilon_2'' \cdot \sin 2(\theta_2'' + i_1 - i_2). \tag{30}$$

The three rotations of the projection, namely (1) rotation about the vertical axis N-S through the angle $2(i_2-i_1)$ with C_1 shifted to C_2' (Fig. 11); (2) rotation about the horizontal axis E-W through the angle ϕ_2 with C_2' shifted to C_2''; (3) rotation about the vertical axis N-S back through the angle $2(i_1-i_2)$ with C_2'' passing to C_2''', are clearly equivalent to a single rotation, through the angle ϕ_2, of the projection about the x_2-axis located at the angular distance $2(i_2-i_1)$ from E, the projection of the x_1-axis.

For the angles θ_2'', θ_2''', ϵ_2'', ϵ_2''', ϕ_2'', ϕ_2''', i_2'', i_2''' other and more complicated expressions can be derived; but to the writer the foregoing procedure and the use of the projection chart in ascertaining the constants of the elliptical vibration that results on the transmission of an incident beam of plane polarized light through one, two, or more birefringent plates appears to have certain advantages, especially that of enabling the observer to visualize the different steps in the process.

By tracing the several angles from one plate to another, as has been done above for the second plate, it is a simple matter for the observer either to write down the equations that are valid for elliptically polarized light, emerging from any one of a set of superimposed plates or to follow in the projection the points that represent the constants of the several successive vibration ellipses. In these operations it is essential that the vibration directions α_n' and γ_n' in each crystal plate be carefully distinguished from the principal axes a_n and b_n of the elliptical vibration of the light waves emerging from the plate.

(5) *Intensity of illumination.* As a special case of the last problem consider the second birefracting plate to be the analyzer. An analyzer consists of calcite and is actually a combination of two birefracting prisms so cut that one of the vibration components, v, (γ'), is diverted by total reflection, thus allowing only the second component, u, (α'), to emerge as a rectilinear or plane polarized beam. No phase difference is introduced by the analyzer. If i' is the angle included between the polarizer and the analyzer, the u_2 component is expressed by 19c except that $\phi_2 = 0$; therefore

$$u_2 = \cos i_1 \cdot \cos (i' - i_1) \cdot \cos \omega t - \sin i_1 \cdot \sin (i' - i_1) \cdot \cos (\omega t - \phi_1) = A' \cdot \cos (\omega t - \phi'). \tag{31}$$

Equating the coefficients of cos ωt and sin ωt in this equation, we find

$$\cos i_1 \cdot \cos (i' - i_1) - \sin i_1 \cdot \sin (i' - i_1) \cdot \cos \phi_1 = A' \cdot \cos \phi' \tag{32a}$$

$$- \sin i_1 \cdot \sin (i' - i_1) \cdot \sin \phi_1 = A' \cdot \sin \phi'. \tag{32b}$$

In these equations A' is the amplitude of the vibration emerging from the analyzer. The intensity of illumination, I, is proportional to the square of the amplitude or to A'^2. On squaring and adding equations 32a, 32b we obtain

$$\cos^2 i_1 \cdot \cos^2 (i' - i_1) + \sin^2 i_1 \cdot \sin^2 (i' - i_1) - \tfrac{1}{2} \cdot \sin 2i_1 \cdot \sin 2(i' - i_1) \cdot \cos \phi_1 = A'^2 = I \tag{33}$$

an expression that reduces to the from

$$I = \tfrac{1}{2} \cdot (1 + \cos 2i_1 \cdot \cos 2(i' - i_1) - \sin 2i_1 \cdot \sin 2(i' - i_1) \cdot \cos \phi_1) \text{ or} \tag{33a}$$

$$I = \cos^2 i' + \sin 2i_1 \cdot \sin 2(i' - i_1) \cdot \sin^2 \frac{\phi_1}{2}. \tag{33b}$$

On substituting, in equation 33a, for $\cos \phi_1$, its equivalent (equation 15) and introducing $\cos 2\epsilon_1$ from (16) we find

$$I = \tfrac{1}{2} \cdot [1 + \cos 2\epsilon_1 \cdot \cos 2(\theta_1 + i' - i_1)]. \tag{34}$$

This equation can be still further simplified if the hypotenuse $2i''$ of a right angle spherical triangle is introduced whose sides are $2\epsilon_1$ and $2(\theta_1 + i' - i_1)$

$$I = \tfrac{1}{2} \cdot (1 + \cos 2i'') = \cos^2 i''. \tag{35}$$

This expression for the intensity depends only on the single angle $2i.''$ Therefore the brightness of all elliptical vibrations represented by points on the small vertical circle, on the sphere, and having $2i''$ as the radius, is the same when examined through an analyzer whose vibration plane includes the angle i'' with the x-axis represented by the point E or W in the projection.

If the nicols are crossed, $i' = 90°$; and equation 33a becomes

$$I = \frac{\sin^2 2i_1}{2}(1 - \cos \phi_1)$$

which is zero for $i_1 = 0$ and also for $i_1 = 90°$; it attains its maximum value for $i_1 = 45°$.

In the projection these relations are easily followed. In Fig. 12 let

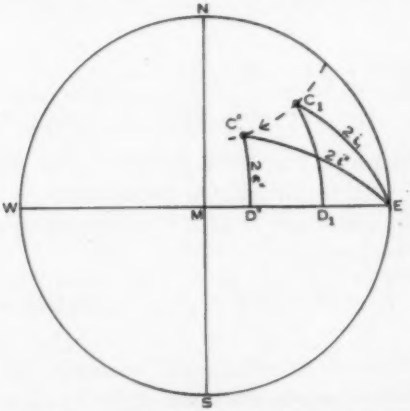


FIG. 12. *In this figure C_1 represents the elliptical vibration emerging from the crystal plate. Refer this vibration to coordinate axes of the analyzer by rotating the projection about the vertical, $N-S$, axis through the angle $2(i'-i_1)$; i' is the angle between the polarizer and the analyzer. The point C_1 is shifted to C''. The angle $2i''$ of the triangle $D''EC''$ meets the requirements of equations 34 and 35 by means of which the intensity can be computed directly.*

C_1 represent the elliptical vibration emerging from the crystal plate (equations 18a, 18b). To find the intensity of the light emerging from the analyzer, whose vibration plane includes an angle i' with that of the polarizer, construct the right angle spherical triangle $C''ED''$ in which $C''D''=C_1D_1=2\epsilon_1$, and $ED''=2(\theta_1+i'-i_1)$. The hypotenuse $2i''$ of this spherical triangle meets the requirements of equations 34 and 35.—If $i'=90°$ and $i_1=0$, $\theta_1=0$, $i''=90°$; the point C'' coincides with W, and the intensity $I=0$. If $i'=90°$ and $i_1=90°$, $\theta_1=90°$, $i''=90°$, C'' coincides with W and the intensity is 0. If $i'=90°$ and $i_1=45°$, $\theta_1=45°$, $i''=\pi-2\epsilon_1$, C'' is on the meridian NWS, the intensity $I=\frac{1}{2}\cdot(1-\cos 2\epsilon_1)=\frac{1}{2}(1-\cos\phi_1)$.

In the case of two superimposed crystal plates considered in problem (4) the final elliptical vibration is represented by the point C_2'' of the spherical triangle $C_2''ED_2''$. If i' is the angle between analyzer and polarizer we can apply, as heretofore, the equation

$$I=\tfrac{1}{2}\cdot(1+\cos 2i_2''\cdot\cos 2(i'-i_2'')-\sin 2i_2''\cdot\sin 2(i'-i_2'')\cdot\cos\phi_2'') \qquad (36a)$$

$$I=\tfrac{1}{2}\cdot[1+\cos 2\epsilon_2''\cdot\cos 2(\theta_2''+i'-i_2'')]=\tfrac{1}{2}\cdot(1+\cos 2i'')=\cos^2 i''. \qquad (36b)$$

For $i'=90°$ this equation reduces to the form

$$I=\frac{\sin 2i_2''}{2}\cdot(1-\cos\phi_2'').$$

The graphical construction in the projection required to ascertain the angle i'', and from it to compute the intensity of the beam of light emerging from the analyzer is identical in method with that described for the single crystal plate.* The analyzer functions actually as an additional crystal that introduces no phase difference ($\phi_a=0$);

* On substituting in equations 36a, 36b the appropriate expressions for $2i_2''$, ϕ_2'', $2\theta_2''$, either from the foregoing equations or by reading them directly from the projection, we obtain the usual expression for the intensity, as given in text books on crystal optics (J. Walker, Analytical Theory of Light), but derived by other methods:

$$I=\cos^2 i'+\sin 2i_1\cdot\sin 2(i_2-i_1)\cdot\cos 2(i'-i_2)\cdot\sin^2\frac{\phi_1}{2}$$

$$+\cos 2i_1\cdot\sin 2(i_2-i_1)\cdot\sin 2(i'-i_2)\cdot\sin^2\frac{\phi_2}{2}$$

$$-\sin 2i_1\cdot\sin^2(i_2-i_1)\cdot\sin 2(i'-i_2)\cdot\sin^2\frac{\phi_1-\phi_2}{2}$$

$$+\sin 2i_1\cdot\cos^2(i_2-i_1)\cdot\sin 2(i'-i_2)\cdot\sin^2\frac{\phi_1+\phi_2}{2}$$

therefore the relations given in equations 20a, 20b suffice to deduce the intensity relations involved in the passage of the component of the elliptical vibration from the last crystal plate through the analyzer.

(5a) For a given elliptical vibration to find the position of the analyzer for which the field brightness is greatest or least, we have from equation 33a

$$\frac{dI}{di'} = \cos 2i_1 \cdot \sin 2(i_1 - i') - \sin 2i_1 \cdot \cos 2(i_1 - i') \cdot \cos \phi = 0$$

or

$$\tan 2(i_1 - i') = \tan 2i_1 \cdot \cos \phi. \tag{37a}$$

In the spherical triangle C_1D_1E (Fig. 12) we have

$$\tan 2\theta_1 = \tan 2i_1 \cdot \cos \phi. \tag{37b}$$

Therefore

$$2(i_1 - i') = 2\theta_1;\ i' = i_1 - \theta_1 \text{ or } i' = \frac{\pi}{2} + (i_1 - \theta_1) \tag{37c}$$

and the position of the analyzer for maximum and minimum extinction is parallel with a or b of the vibration ellipse. For these positions, (equations 33a, 13)

$$I = \tfrac{1}{2} \cdot (1 \pm \cos 2i_1 \cdot \cos 2\theta_1 + \sin 2i_1 \cdot \sin 2\theta_1 \cdot \cos \phi_1)$$
$$= \tfrac{1}{2} \cdot (1 \pm \cos 2\epsilon_1) = \cos^2\epsilon_1 \text{ or } \sin^2\epsilon_1.$$

depending on the sign within the parenthesis.

If the analyzer plane is parallel either with a or b a rotation through a given angle either to the right or left produces the same change in field brightness. If, therefore, a biquartz rotatory wedge or plate be used with the analyzer or a split-nicol alone, the line between matched fields is the direction of either a or b of the vibration ellipse. The two positions of the analyzer for matched fields are 90° apart. The angle i_1 is the angle between the polarizer plane and α' of the crystal plate; i' is the angle between the polarizer and the analyzer. The measurement of matched fields furnishes i_1 and i'; but $i_1 - i' = \theta_1$. Therefore in the spherical triangle $2i_1$ and $2\theta_1$ are given; these suffice for the computation of $2\epsilon_1$ and ϕ_1.

(5b). In case the biquartz rotatory plate or the split-nicol is inserted between the polarizer and the crystal plate, as in the method of Sénarmont, the expression for the intensity becomes for the matched fields, respectively, if $\pm\rho$ is the small rotation angle of the rotatory plate, (33b)

$$I = \cos^2(i' + \rho) - \sin 2(i_1 + \rho) \cdot \sin 2(i_1 - i') \cdot \sin^2 \frac{\phi}{2} \qquad (38a)$$

$$I = \cos^2(i' - \rho) - \sin 2(i_1 - \rho) \cdot \sin 2(i_1 - i') \cdot \sin^2 \frac{\phi}{2} \qquad (38b)$$

Equating these expressions we find

$$\tan 2i_1 = \tan 2(i_1 - i') \cdot \cos \phi_1 \qquad (39)$$

which is satisfied for the two positions, $(i_1' - i')$ and $[(\pi/2) + i_1 - i']$. If the light were allowed to traverse the combination in the reverse direction, equations 37a to c would apply. In this case i_1 becomes $i_1 - i'$ while i' becomes $-i'$; with these changes equation 39 is identical with (37a) and becomes understandable. For either position of the biquartz plate the measurements yield i_1 and θ_1, from which in turn ϵ_1 and ϕ_1 can be computed.

(6) *Measurement of the azimuth and ellipticity of an elliptical vibration.* The simplest, but not the most accurate, method for measuring the constants of an incoming elliptical vibration is to use a quarter wave plate or a compensator set at the phase difference $\pi/2$, (retardation $\frac{1}{4}$ wave), and an analyzer and to adjust the positions of the two until the field of view is dark. Equations 4 and 5 show that, in case the ellipse is referred to its principal axes a, b, the phase angle is $\pi/2$ or $(4n+1) \cdot \pi/2$ when the movement of the vector is counterclockwise; and $-\pi/2$ or $(4n-1) \cdot (\pi/2)$, when the movement is clockwise (Figs. 1, 3a, 3b). If a phase angle $\pi/2$ be added to, or subtracted from the phase angle $\pm\pi/2$, the resulting phase angle is π or 0 or multiples thereof; the emergent vibration is rectilinear and can be extinguished by an analyzer set at the proper angle. The directions of the principal ellipse axes in the quarter wave plate are known; therefore the sense of the movement of the vector of the incoming vector movement, whether laevogyre or dextrogyre, can be determined. At each one of the four possible positions of coincidence of the principal axes of the quarter-wave plate with the principal axes of the elliptical vibration, the resulting vibration is rectilinear. By measuring the angle between two successive positions of extinction, the angle 2ϵ or $2(\pi - \epsilon)$ is ascertained and with it the ellipticity $\tan \epsilon = \pm(b/a)$. The position of the principal ellipse axis α' in the quarter-wave plate when extinction is obtained, marks also the position of one of the ellipse axes of the elliptical vibration and with it the angle θ. In case the major axis α' of the quarter-wave plate coincides with the major axis a_1 of the elliptical vibration under test, the resultant phase angle is π and the

rectilinear vibration is along OV' (Fig. 1, but modified so that the x, y axes coincide with the principal ellipse axes). If now the quarter-wave plate be turned 90° so that its α' axis coincides with the shorter, b, axis of the elliptical vibration, the resultant phase angle is zero and the rectilinear vibration is along OV. The angle of counter clockwise rotation of the analyzer from the first to the second extinction position is less than 90° in the case of laevogyre elliptical vibration.

In the projection the phase angle, ϕ, for all points on the equator is zero or π or multiples thereof, the ellipticity is zero ($2\epsilon=0$), and the vibration is rectilinear. For all points on the meridian $ENWS$ the phase angle is $\phi=\pm\pi/2$ or odd multiples thereof; the components of the

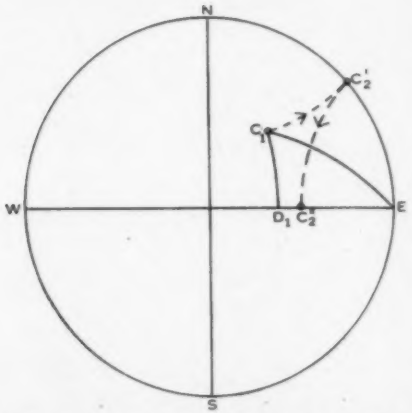


FIG. 13. *The point C_1 represents the incoming elliptic vibration. Refer this to the axes of the greater wave plate by rotating the projection about the vertical axis through angle $2(i_2-i_1)=-2\theta_1$; C_1 is shifted to C_2'; rotate projection about horizontal, $E-W$, axis through phase angle $\pm\pi/2$ of quarter-wave plate and shift C_2' to $-C_2''$ or to $+C_2''$, depending on the position of the quarter-wave plate. C_2'' is on the equator and represents a rectilinear vibration.*

vibration are along the principal axes of the ellipse. Let the point C_1 (Fig. 13) define the incoming elliptical vibration. The quarter-wave plate functions as a second superimposed plate and equations 21 and 22 apply to it. In this case, however, $\phi_2=\pm\pi/2$ and the equations reduce to the form

$$\sin 2\epsilon_2'' = \cos 2\epsilon_1 \cdot \sin 2(\theta_1 + i_2 - i_1) \tag{40}$$

$$\tan \theta_2'' = \mp \frac{\tan 2\epsilon_1}{\cos 2(\theta_1 + i_2 - i_1)}. \tag{41}$$

If, in addition, the transformed vibration is to be rectilinear,

$$\sin 2\epsilon_2'' = 0.$$

Therefore

$$2(\theta_1 + i_2 - i_1) = 0 \text{ or}$$

$$i_2 = i_1 - \theta_1 \text{ or}$$

$$i_2 - i_1 = -\theta_1$$

also

$$\theta_2'' = -\epsilon_1.$$

On rotating the projection about the vertical axis through the angle $2(i_2-i_1) = -2\theta_1$, the point C_1 is shifted along the small horizontal circle to C_2' on the meridian, $N-E$, for which the phase angle is $\pi/2$ and the ellipticity is $\tan \frac{1}{2} C_2'E = b/a$. The phase angle, $\phi_1 = \pi/2$, is now combined with that of the quarter-wave plate by rotation about the horizontal axis through $\pm 90°$, thus shifting C_2' to C_2'' on the equator. The resultant phase angle is π or 0. Angle $C_2''E = C_2'E = C_1D_1 = 2\epsilon_1$. The ellipticity is $\tan \epsilon_1$ and the azimuth θ_1 is half the angle ED_1. For the phase angle $\phi_2 = -\pi/2$, C_2' is shifted in the reverse direction to the point C_2'' in the positive direction of the equator, and $\theta_2'' = \epsilon_1$. If ϕ_1 is between 0 and π and 2π, the direction of movement of the vector is counterclockwise for an observer facing the incoming stream; for ϕ_1 between π and 2π it is clockwise.

In this measurement it is difficult to determine the exact position of extinction. If instead of the analyzer alone a biquartz rotating wedge[2] is placed in front of the analyzer, or a split-nicol is substituted for the analyzer, a photometric field is obtained and the accuracy of setting for extinction is increased (equations 38a, 38b).

(6a) *Friedel's method.* In this method a quarter-wave plate is used in front of the crystal plate and in the diagonal position with it. The analyzer plane coincides with the γ' direction of the quarter-wave plate. The polarizer is turned to the position of extinction of field illumination. The phase angle of the crystal plate is then half the angle included between the vibration plane of the polarizer and the α' direction of the quarter-wave plate. In the projection (Fig. 14) let C_1 represent the quarter-wave plate and C_2 the crystal plate whose phase angle is $-\phi_2$. The angle $C_1C_2' = 90°$. In the projection the point C_1 is first shifted by rotation through the angle $2(i_2-i_1) = 90°$ about the vertical axis to the central meridian NMS to C_2' and then by rotation about the

[2] F. E. Wright, Am. J. Sci. (4) *26*, 377–378; 1908.

horizontal axis E-W through the angle $2i_1 = -\phi_2$ to C_2'' on the equator; a second rotation about the vertical axis carries C_2'' to E, the x_1 axis. The resultant vibration is rectilinear, the vibration plane of the analyzer is at W, the γ' axis of the quarter-wave plate.

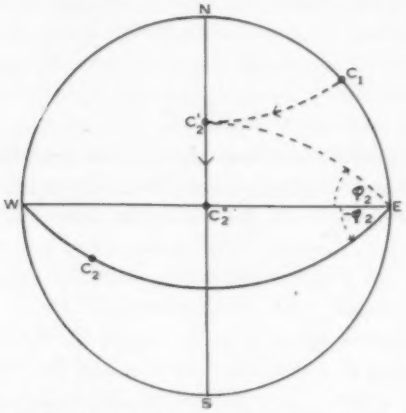


FIG. 14. *To refer the vibration from the quarter-wave plate (C_1) to the axes of the crystal plate rotate the projection about the vertical axis through $2(i_2-i_1)=90°$ and shift C_1 to C_2'. Rotate projection about the horizontal, $E-W$ axis through the phase angle, $-\phi_2$, and shift C_2' to C_2'' on the equator. C_2'' is then shifted to E by rotation of the projection about the vertical axis through he angle $2(i_1-i_2)$.*

(6b) The *Bravais* bi-plate consists of two quarter-wave plates placed side by side with a sharp line of separation and with α_1' of the first plate parallel with γ_2' of the second. The first quarter-wave plate introduces a phase change of $+\pi/2$ and the second a phase change of $-\pi/2$. If the ellipse axes of the quarter-wave plates are placed parallel with the principal axes of the elliptical vibration (the point C is on the meridian NES in the projection), the resulting vibration is rectilinear; the phase angle in the one half is zero; in the second half, π. The two halves of the field are equally bright only when the transformed vibration is rectilinear.

(7) Given an elliptical vibration and a crystal plate of phase angle ϕ. How should the plate be placed to reduce the elliptical vibration to a rectilinear movement? In this case $\epsilon_2''=0$ and equations 21 and 22 reduce to the form

$$\sin 2(\theta_1 + i_2 - i_1) = \mp \tan 2\epsilon_1 \cdot \cot \phi_2 \tag{42}$$

$$\cot 2\theta_2'' = \cot 2(\theta_1 + i_2 - i_1) \cdot \cos \phi_2. \tag{43}$$

From equation 42 the angle i_2 between the polarizer and α_1' of the crystal plate can be computed; and from equation 43, $2\theta_2''$, the azimuth of the resulting rectilinear vibration.

In the projection (Fig. 15) let C_1 represent the incoming elliptical vibration and C_2 that of the crystal plate. In this plate the phase angle $-\phi_2$ remains constant. To offset exactly the phase angle of the incoming elliptical vibration it is necessary to refer the vibration to a set of coordinate axes such that the new phase angle $\phi_2' = -\phi_2$ or $\pi - \phi_2$. To do this rotate the projection about the $N-S$ axis and shift the point C_1 to C_2', the intersection of the small horizontal circle C_1C_2' with the arc EC_2' which includes an angle $+\phi_2$, with the equator. The azimuth

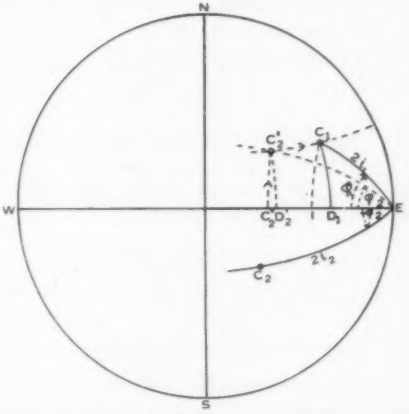


FIG. 15. *The points C_2, C_2'', and C_1 represent, respectively, the elliptical vibration from the compensator, the restored rectilinear vibration and the unknown elliptic vibration. To find the position of C_1 the projection is rotated about the horizontal, $E-W$, axis through the phase angle, ϕ_2, and C_2'' is shifted to C_2'. The point C_1 lies on the small circle $C_2'C_1$. The azimuth of C_2' is $2\theta_2'' = 2(\theta_1 + i_2 - i_1)$; therefore $2(i_2 - \theta_2'') = 2(i_1 - \theta_1)$. On the small circle $C_2'C_1$ locate the point C_1 for which $2(i_1 - \theta_1) = 2(i_2 - \theta_2'')$ and with it determine $2i_1$ and ϕ_1.*

angle ED_2' is $2\theta_2' = 2(\theta_1 + i_2 - i_1)$. From the triangle $EC_2'D_2'$ equations 42 and 43 can be read off directly, if we note that the position angle, $EC_2' = 2i_2'$, becomes, on rotation of the projection about the horizontal axis $E-W$ through the angle, $-\phi_2$, the azimuth angle $2\theta_2''$.

The effect of the crystal plate is to introduce the phase angle, ϕ_2, which is either the supplement of the angle ϕ_2', (C'_2 in the upper hemisphere), or it is equal and opposite to ϕ_2, ($\phi_2 = -\phi_2'$; C_2' in the lower hemisphere). The resultant phase angle is either π or zero and the transformed vibration is rectilinear.

(7a) The converse problem is easily solved in the projection. An unknown ellipse, C_1, is restored by a compensator of phase angle ϕ_2 and azimuth θ_2 to a rectilinear vibration of azimuth θ_2''; to determine the ellipticity, tan ϵ_1, and the azimuth, θ_1, of C_1. In the projection (Fig. 15) the points C_2 and C_2'' are given; C_2 represents the compensator and C_2'', the transformed rectilinear vibration. With E representing the direction α_2' of the compensator introduce the phase angle ϕ_2 by rotating the projection about the horizontal axis $E-W$ through $+\phi_2$ and transfer C_2'' to C_2'. The azimuth of C_2' is $\theta''_2=\theta_1+i_2-i_1$ or $i_2-\theta_2''=i_1-\theta_1$. The point C_1 is on the small circle $C_2'C_1$ distant $2\epsilon_1$ from the equator. On this circle locate by trial the point C_1 for which $2i_1-2\theta_1$ is equal to the known angle $2(i_2-\theta_2'')$, and with it determine the phase angle ϕ_1 and the azimuth $2\theta_1$.

In triangle $C_2'D_2'E$ we have the relations

$$\sin 2\epsilon_2' = \sin 2i_2'' \cdot \sin \phi_2$$

$$\tan 2\theta_2'' = \tan 2i_2'' \cdot \cos \phi_2$$

in which the angles $2i_2''$ and ϕ_2 are given, and from which $2\epsilon_2'$ and $2\theta_2''$ can be computed; also the difference S

$$2i_2 - 2\theta''_2 = 2(i_1 - \theta_1) = S. \tag{44}$$

In triangle C_1D_1E we have the relation

$$\cos 2i_1 = \cos 2\epsilon_1 \cdot \cos 2\theta_1 \tag{44a}$$

in which $2\epsilon_1=2\epsilon_2'$ and from equation (44), $2i_1=S+2\theta_1$. Substituting these values in equation (44a) we obtain the relation

$$\tan 2\theta_1 = \frac{\cos S - \cos 2\epsilon_2'}{\sin S} \tag{45}$$

and thus find $2\theta_1$. The angles $2\theta_1$ and $2\epsilon_1$ suffice to locate the point C_1 and with it the elements of the triangle C_1D_1E.

(7b). The Stokes[3] method for ascertaining the phase angle and the ellipticity of an incoming elliptic vibration is based essentially on this method. The compensator of phase angle slightly different from 90° and the analyzer are each mounted in a support about an axis in the line of sight. Equations 14 and 17

$$\sin 2i = \frac{\sin 2\epsilon}{\sin \phi} \tag{14}$$

[3] C. G. Stokes, Phil. Mag. (4), *2*, 420; 1851.

$$\sin 2\theta = \tan 2\epsilon \cdot \cot \phi \tag{17}$$

are satisfied, for a given 2ϵ, for the values, respectively, i, $(\pi/2)-i$, $(\pi/2)+i$, $\pi-i$, and θ, $(\pi/2)-\theta$, $(\pi/2)+\theta$, $\pi-\theta$, if proper regard for the sign of ϕ is taken in each case. For any two successive values we have

$$i_1 + i_2 = \frac{\pi}{2} \quad \text{and} \quad \theta_1 + \theta_2 = \frac{\pi}{2}.$$

For these values equations 15 and 16 become

$$\cos \phi = \frac{\tan 2\theta_1}{\tan 2i_1} = \frac{\tan 2\theta_2}{\tan 2i_2} = \frac{\cot (\theta_2 - \theta_1)}{\cot (i_2 - i_1)} \tag{46}$$

$$\cos 2\epsilon = \frac{\cos 2i_1}{\cos 2\theta_1} = \frac{\cos 2i_2}{\cos 2\theta_2} = \frac{\sin (i_2 - i_1)}{\sin (\theta_2 - \theta_1)} \tag{47}$$

from which the phase angle ϕ and the ellipticity can be computed. The angles θ_1, θ_2 are the readings of the positions of the compensator; i_1, i_2 are the corresponding analyzer positions for two successive extinction positions.

(8) Determine the phase angle of a crystal plate by measuring its ellipticity. Equation 14 shows that if $2i_1 = 90°$, or, if the given plate is placed in the diagonal position between crossed nicols, the phase angle $\phi_1 = 2\epsilon_1$; equation 13 shows that in this case, $2\vartheta_1 = 90°$. With the crystal plate in this position superimpose a quarter-wave plate and determine the positions both of it and of the analyzer for which extinction is obtained. The angle between the α' direction of the quarter-wave plate and the plane of the analyzer is the complement both of the angle, ϵ_1, or $\pi-\epsilon_1$, and of the angle $\phi_1/2$. If in equations 21 and 22 the values $2i_1 = 2\vartheta_1 = 90°$ and $\phi_2 = 90°$ be inserted we find that for the resultant vibration to be rectilinear or $\epsilon_2'' = 0$, $i_2 = 0$ and $\theta_2'' = \epsilon_1$. In the projection (Fig. 16) let C_1, on the central meridian NMS represent the elliptical vibration emerging from the crystal plate; $EC_1 = 2i_1 = 90° = 2\theta_1$ and $C_1M = 2\epsilon_1 = \phi_1$. For C_2 representing the quarter-wave plate, $i_2 = 0$, $\phi_2 = 90°$, and C_2 coincides with E. The point C_1 is shifted to C_2' on the meridian NES. The effect of the quarter-wave plate is represented by a rotation of the projection about the horizontal axis $E-W$ through 90°, whereby C_2' is moved to C_2'' on the equator. The phase angle $\phi_1 = C_2''E = C_2'E = C_1M$. If $\phi_2 = +\pi/2$, the point C_2'' is on the negative side of E and on the

equator; if $\phi_2 = -\pi/2$, C_2'' is to the left of E and in the positive direction from E along the equator.

In the diagonal position between crossed nicols, the two components of the light vector movement along α' and γ', respectively, of the crystal plate are

$$x = \cos\frac{\phi}{2}\cdot\cos\omega t$$

$$y = \sin\frac{\phi}{2}\cdot\sin\omega t.$$

(9) In the photo-elastic analysis of stress-strain relations in a trans-

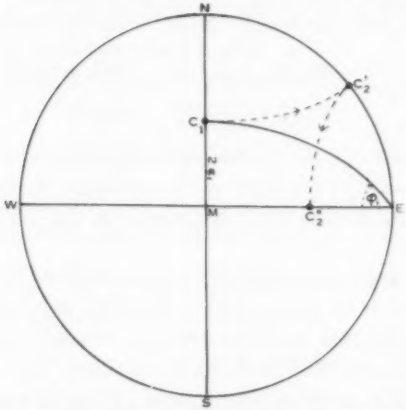


FIG. 16. *The point C_1 represents the vibration from the crystal plate; C_2' this vibration referred to the axes of the quarter-wave plate; C_2'', the vibration after the phase angle, $-\phi_2$, of the quarter-wave plate has been introduced. The angle EC_2'' is the phase angle, ϕ_1, of the crystal plate*

parent plate under load or in the observation of the isochromatic lines in an interference figure produced in convergent polarized light by a birefracting plate, it may be convenient to eliminate the zero isogyres or interference brushes and to have all points of an isochromatic line at the same intensity level. This can be accomplished by the method first used by Maxwell, namely, to view the plate in circularly polarized light. Circularly polarized light is produced by inserting a quarter-wave plate (phase angle = 90°) directly above the polarizer and at an azimuth of 45° with its vibration plane; a second quarter-wave plate is placed directly below the analyzer and at an azimuth of 45° with its vibration plane. The birefracting plate is inserted between the two quarter-wave plates.

For the first quarter-wave plate we have, $\phi_1 = \pm 90°$, $2i_1 = 90°$, $2\epsilon_1 = 90°$, $2\theta_1 = 90°$; and for the crystal plate, ϕ_2, $2i_2$, $2\epsilon_2$, $2\theta_2$. Setting these values in equations 21 and 22, we find $2\epsilon_2'' = 90° \pm \phi_2$, $2\theta_2'' = \mp 90°$, $2i_2'' = \mp 90°$, $\phi_2'' = 90° \pm \phi_2$.

In the projection (Fig. 17), for all points on the great circle *baE*, the phase angle, ϕ, is constant; in other words any crystal plate of phase angle, ϕ, whose α'-axis may include any angle i with the polarizer, is represented by a point on *WbaE*. If now the vibration ellipse represented by C_2' be referred to the original x_1, y_1 axes of the quarter-wave plate, the projection is rotated about the vertical axis $N-S$ and the point C_2' representing any given point of the great

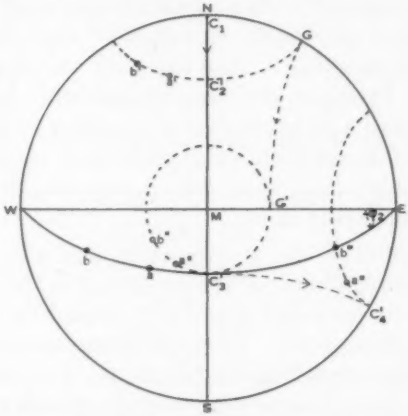


FIG. 17. *The point C_1 represents the circular vibration emerging from the first quarter-wave plate; it can be referred to any set of coordinate axes without change in position. Introduce phase angle, $-\phi_2$, of any point on great circle EabW by shifting C_1 to C_2'. To refer this vibration to the original x_1, y_1 axes rotate the projection about the vertical axis through angle $2(i_1 - i_2)$ and shift C_2' to a' or some other point on the small circle $C_2'G$. If the axes of the second quarter-wave plate coincide with x_1, y_1, the phase angle, $\pm\pi/2$, is introduced by rotating the projection about the horizontal, $E-W$, axis through $\pm\pi/2$. If the phase angle is $-\pi/2$, the pole N is shifted to M at the midpoint of the projection and the small circle $C_2'G$ to the small circle $C_3'G'$ concentric about M. To refer any point on the small circle $C_3'G'$ to the coordinate axes of the analyzer, distant 45° from the x_1, y_1 axes, rotate the projection about the vertical axis through the angle, $2 \times 45° = 90°$, and shift the small circle $C_3'G'$ to the small circle $C_4'a'''b'''$ concentric about E. All points on this small circle have the same intensity of illumination.*

circle *Eab* is shifted along the small, horizontal circle $C_2'G$. For each point, $2\theta_2''' = \pm 2i_2$, $2\epsilon_2''' = 90° \pm \phi_2$; the values i_2''' and ϕ_2''' can be computed by use of equations 29 and 30. The elliptically polarized light now enters the upper quarter-wave plate. If its axes

of reference coincide with those of the first quarter-wave plate (x_1, y_1) no rotation about the vertical is needed; only the rotation through $\pm 90°$ about the horizontal axis $E-W$, to introduce the phase angle of the quarter-wave plate. By this rotation the pole N is brought to coincide with M, the central point in the projection. The small circle $C_2'G$ about N becomes on rotation the small circle $C_3'G'$ concentric about M and having the angular radius ϕ_2. The light now enters the analyzer whose vibration plane includes an angle of 45° with α' of the upper quarter-wave plate. No phase angle is hereby introduced; only a rotation of the projection about the vertical $N-S$ axis through 90°, whereby the point M is brought to coincidence with E and the small circle $C_3'G'$ to coincide with the small circle $C_4'a'''b'''$, concentric about E and having the radius $2i^{\text{IV}} = \phi_2$. But all points on a vertical small circle about E have the same intensity of illumination (equation 35). Therefore with this arrangement the intensity of illumination remains unchanged during a rotation of the crystal plate. The intensity of illumination for a given point is, moreover, the intensity which is obtained when the crystal plate is in the diagonal position between crossed nicols.

The combination of the quarter-wave plate with analyzer in the diagonal position is shown in the projection to be equivalent to a rotation about the horizontal axis $E-W$ through 90° and then about the vertical axis through 90°. These two operations are equivalent to a single rotation through 90° about the axis M. This means that, as the combination of quarter-wave plate and analyzer is turned, its effect remains the same; the intensity of illumination of a point in the field remains unchanged on rotation of the quarter-wave plate, analyzer combination. In an interference figure, the zero isogyres or brushes are no longer present and the courses of the isochromatic lines are seen most distinctly.

(9a). In this connection an interesting experiment of Fresnel may be given to illustrate the projection method. Fresnel superimposed two quarter-wave plates in crossed position so that α_1' was parallel with γ_2'; between the two in diagonal position he inserted a crystal plate. He proved that with this combination the emergent light was plane polarized; also that the angle between the analyzer, set at extinction, and the polarizer is half the phase angle of the crystal plate, irrespective of the azimuth of the polarizer. This conclusion is readily deduced from the equations, but also by use of the projection. In Fig. 18 let C_1, C_2,

C_3 represent, with respect to the polarizer, the positions, respectively, of the combination of quarter-wave plate, α_1', crystal plate, α_2', (angle α_1': $\alpha_2' = 45°$), and quarter-wave plate, α_3', (angle α_1': $\alpha_3' = 90°$). For this combination the angle between the polarizer plane and the plane of the emergent plane-polarized beam, C_3^{iv} is half the phase angle ϕ_2; the angle between C_3^{iv} and E in the projection is equal to the phase angle ϕ_2. But this projection angle is twice the actual angle; therefore i', the angle between the polarizer and the analyzer, is equal to half the phase angle of the crystal plate.

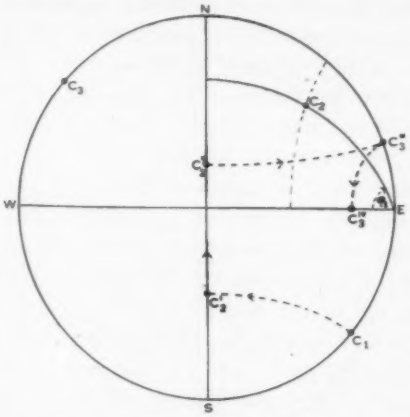


FIG. 18. *The points C_1, C_2 and C_3 represent the elliptical vibrations of the first quarter-wave plate, the crystal plate, and the second quarter-wave plate respectively. Refer the elliptical vibration C_1 to the C_2 axes by shifting C_1 to C_2'; introduce the phase angle, ϕ_2, by shifting C_2' to C_2''. Refer the vibration C_2'' to the axes of the second quarter-wave plate by shifting C_2'' to C_3''' and introduce the phase angle, $-\pi/2$, by shifting C_3''' to C_3^{iv} on the equator. The angle EC_3^{iv} is equal to ϕ_2 and is twice the angle between the polarizer and analyzer.*

(10) The Brace[4] method for the analysis of incoming elliptically polarized light uses, in addition to a compensator and an analyzer, each separately rotatable about an axis in the line of sight as in the Stokes method, a very thin mica strip; it has a very small phase angle, and is attached at a fixed angle, usually 45°, to the analyzer. This mica strip covers only half the field. The compensator and analyzer are turned to positions for which the two halves of the field are equally bright. From the setting angles of two successive positions of matched

[4] Brace, D. B. Half shade elliptical polarizer and compensator. Phys. Review (I) *18*, 70 1901; (I) *19*, 218, 1902.

fields, the characteristics of the elliptical vibration can be computed. In the projection the relations are readily followed (Fig. 19). Let C_1 represent the incoming vibration, whose elements are ϕ_1, ϵ_1, i_1, and θ_1; the phase angle of the compensator is $-\phi_2$; that of the thin mica half-shade, $-\phi_3$. At the position of equally bright fields, the plane of the analyzer is equally distant from the principal axes a_3' and a_3'' of the elliptical vibrations producing the two fields (equations 38a, 38b). The function of the half shade plate, whose α' ellipse axis is commonly 45° distant from the analyzer plane, is to introduce a phase change, such that the resulting phase angle of its field is equal and opposite to that

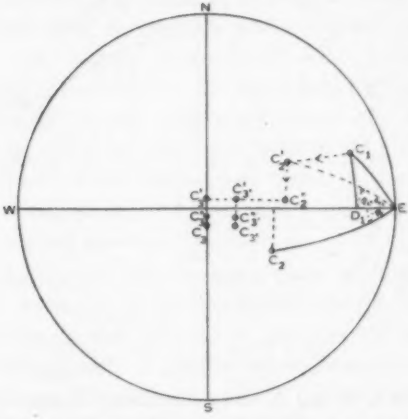


FIG. 19. *The point C_1 represents the incoming elliptical vibration; C_2, the elliptical vibration of the compensator and C_2 (or C_2') that of the thin mica strip attached at a definite angle to the analyzer and covering only a portion of the field. The point C_3' (or C'_3) represents the elliptical vibration emerging from the uncovered portion of the field; C_3'' (or C''_3) the elliptical vibration from the covered part. The angle between C_3' and the equator is $\phi_2/2$; that between C_3'' and the equator is $-\phi_2/2$. To find the position of C_2, shift C_1 to the point C_3' for which the phase angle is $\phi_3+\phi_2/2$. The angle of rotation $2(i_2-i_1)$ on this shift can be read off directly in the projection and the position angle $2i_2$ thus ascertained and with it the position of C_2 on the great circle of phase angle $-\phi_2$. Shift C_3' to C_3'' and introduce phase angle $-\phi_2$. Refer the vibration C_3'' to the axes of the mica strip by shifting C_3'' to C_3' (or C'_3 if the angle between α' of the mica strip and the analyzer is different from 45°). The point C_3' represents the vibration from the uncovered portion of the field. To find the vibration from the covered portion shift C_3' to C_3'' through the phase angle $-\phi_2$. Points C_3' and C_3'' are on the same meridian, on opposite sides of, and equidistant from, the equator; the two fields are therefore equally bright.*

of the phase angle of the adjacent field. The phase angle of the one field is $+\phi_3/2$; that of the second, $-\phi_3/2$. The two points C_3' and C_3'' in the projection, representing the two equally bright fields are therefore

equally distant from the equator and from E. The problem in the projection is accordingly that of finding the position of C_2, which represents the compensator of phase angle $-\phi_2$, which shall shift the point C_1 to a point C_2'' whose phase angle is $+\phi_2/2$. The first operation is to rotate the projection about the vertical axis through an angle that shifts C_1 to C_2' on the great circle EC_2', which includes with EW the angle $\phi_2+(\phi_2/2)$; the angle of rotation, i_2-i_1, serves to determine i_2 and with it the position of the compensator represented by C_2 on the great circle $-\phi_2$. By this rotation the incoming vibration is referred to the coordinate axes of C_2. The phase change is accomplished by rotation about the horizontal axis $E-W$, through the angle $-\phi_2$; C_2' is shifted to C_2'' for which the phase angle is $+\phi_2/2$. To refer now this vibration C_2'' to the analyzer plane, rotate the projection about the vertical axis and shift C_2'' to C_3' on the meridian on which C_3 is located; in this position $C_3'D_3=C_3''D_3$ and the fields are equally bright.

(10a). The converse problem is similar to that of problem (7a). An unknown elliptical vibration C_1 is restored by a compensator of phase angle ϕ_2 and azimuth θ_2 to a vibration of known azimuth θ_2'' and phase angle $+\phi_2/2$; to determine the ellipticity and phase angle of C_1. The given data suffice to locate C_2 and C_2'' in the projection. The first step is to rotate the projection about the horizontal axis, $E-W$, through the angle ϕ_2 and shift C_2'' to C_2' at the azimuth angle $2\phi_2'$; the angle $C_2'D_2'=2\epsilon_2'$. The second step is to rotate the projection about the vertical axis, and shift C_2' to C_1 for which $i_1-\theta_1=i_2-\theta_2'$. This position can be found by trial in the projection; or the angle θ_1 can be computed, as in section (7a). The point C_2'' represents the transformed elliptical vibration of known phase angle $\phi_2/2$ and azimuth θ_2''; the position angle i_2'' can be computed by use of equation 15

$$\cot 2i_2'' = \cot 2\theta_2'' \cdot \cos\frac{\phi_2}{2}.$$

In the spherical triangle $C_2'D_2'E$ we have (equations 14 and 15)

$$\sin 2\epsilon_2'' = \sin 2i_2'' \cdot \sin\left(\phi_2 + \frac{\phi_2}{2}\right)$$

$$\tan 2\theta_2'' = \tan 2i_2'' \cdot \cos\left(\phi_2 + \frac{\phi_2}{2}\right)$$

in which the angles $2i_2''$ and $(\phi_2+\phi_2/2))$ are given; the angles $2\epsilon_2''$ and $2\phi_2''$ and also the difference S

$$2i_2 - 2\theta_2'' = 2(i_1 - \theta_1) = S \tag{48}$$

can be computed. In triangle C_1D_1E in turn equation 16 reads

$$\cos 2i_1 = \cos 2\epsilon_1 \cdot \cos 2\theta_1$$

in which $2\epsilon_1 = 2\epsilon_2'$ and $2i_1 = S + 2\theta_1$. Therefore

$$\cos (S + 2\theta_1) = \cos 2\epsilon_2' \cdot \cos 2\theta_1 \qquad \text{or}$$

$$\tan 2\theta_1 = \frac{\cos S - \cos 2\epsilon_2'}{\sin S}. \tag{49}$$

The angle $2\theta_1$ computed from equation 49, together with $2\epsilon_1 = 2\epsilon_2'$ suffice to locate the point C_1 and with it the characteristics of the elliptical vibration under test. The phase angle ϕ_1 can be read off directly from the projection or computed by use of equation 17.

(10b). The introduction by Tool[5] of a split nicol in the Brace apparatus in place of the analyzer furnishes a field divided into quadrants and increases the accuracy with which settings can be made. The effect of the split nicol or of a bi-quartz rotatory plate or wedge is discussed in Section 5a.

Additional problems might be cited to illustrate further this method of treatment and to deduce the relations adopted in the methods used to ascertain the constants of a beam of elliptically polarized light; but enough has been given to demonstrate the principles involved. The use of the projection chart enables the observer to visualize the geometric relations underlying elliptical vibration problems, to solve these problems graphically and also to write down directly the formulas required in the analytical solutions. In general these formulas are long and complicated; unless they are developed and checked by reference to the spatial relations as expressed in a projection chart, there is danger that the physical significance of the formulas may be overlooked. If the process is, however, followed step by step in the projection and the appropriate formulas are read off directly therefrom, not only are the formulas simpler, but there is less chance for error in the computations.

[5] A. Q. Tool, Phys. Rev., *31*, 1; 1910.

THE REFLECTION AND TRANSMISSION OF LIGHT BY PHOTOGRAPHIC PLATES

By D. Brent McRae and Richard C. Tolman
[California Institute of Technology, Pasadena, California. Received July 7, 1930]

Introduction

A knowledge of the extent to which light is transmitted and reflected by a photographic plate is of importance for consideration of the light energy required to render developable a grain of silver bromide, as well as for certain practical problems. Unfortunately, such information is meager and difficult to find. Nutting[1] determined the diffuse transmission and reflection of several types of commercial plates by a visual method using white light. Eggert and Noddack[2] used an apparatus lined with thermocouples similar to an integrating cube to determine the total transmission and reflection of plates for various wave lengths of monochromatic light. Their work was confined, however, to two types of German plates. Helmick[3] made rough estimates of the behavior toward ultraviolet light of an unspecified commercial emulsion recoated on quartz plates. Huse is reported by Ross[4] as having determined the transmissions of a number of commercially available plates for white, red, green, and blue light. His method was photographic one but details are not available.

Because of the lack of extensive investigations in this field, and because precautions were evidently not always taken in the foregoing work to prevent overexposure during the process of making the measurements, it was thought worth while to determine the transmission and reflection of commercial plates under the conditions of ordinary photographic use. A complete series of measurements were carried out on fresh, unexposed Eastman 40 plates, emulsion no. 2637. These were chosen as a typical example of a widely used, clean working, commercial plate giving reproducible results. A photographic method was used in order to permit measurements to be made without exposing the plate to so much light as to change its optical properties, which would be the case with a visual method.

[1] P. G. Nutting, Trans. Ill. Eng. Soc., *10*, 353; 1915.
[2] Eggert and Noddack, Zeit. f. Phys., *20*, 299; 1923.
[3] P. S. Helmick, J. Opt. Soc. Am., *9*, 521; 1924.
[4] F. E. Ross, "*The Physics of the Developed Photographic Image*," page 40 (Van Nostrand)

LIGHT SOURCE AND PHOTOMETER

The light source was a 3.8 volt flashlight lamp mounted in a metal lamp house, and operated at slightly less than normal brilliancy. A diaphragm in contact with ground glass served to adjust the intensity of the illumination. The filament current was supplied from storage cells and was held constant at 0.280 amperes by means of a rheostat and milliammeter. The visual candle power of the lamp with ground glass and diaphragm in place was measured on a bench photometer and found to be 0.158 c. p. The distance from diaphragm to plate was 157 cm in the transmission experiments and 39 cm in the reflection experiments.

Measurements and comparisons of the densities of optical wedges and plates were made by means of a Marten's type polarization photometer, the density to be determined being placed in contact with an illuminated opal glass plate. A series of supplementary densities ranging from 0.630 to 3.05 was used for the measurement of all densities greater than 1.0 to avoid trouble from scattering, and to avoid the use of the higher portions of the photometer scale.

TRANSMISSION MEASUREMENTS

Fig. 1 shows diagrammatically the method of measuring transmissions. An unexposed plate (the receiver plate) was laid face up on

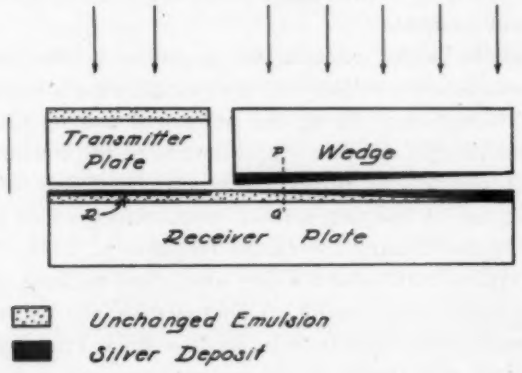


FIG. 1.

the bench under the light source. Over one side was laid a strip of the plate to be tested (the transmitter plate) also face up. Over the other side was placed a strip of optical wedge. After allowing light to fall on this assembly, the lower or receiver plate was developed. A point,

Q, on the wedge image was then picked out which had the same density as some average point, R, under the transmitter plate. The transmission of the wedge was then measured at a point P, on the wedge, corresponding to Q, and this value was taken to represent the transmission of the transmitter plate.

A minor departure from the conditions of the ordinary use of photographic plates should be mentioned. In a camera the light falling on the plate at any given point is nearly parallel. In these experiments, however, the incident light was diffused by a sheet of ground glass which was purposely placed a few centimeters in front of the plate. This appeared to be desirable for two reasons:

1. The measurement of the density of the wedge is made in the photometer by the use of diffused light, consequently exposure should be made to the same type of light. 2. The light falling on the receiver plate at R has been diffused by passage thru the transmitter plate, so the deposit at Q, should also be produced by diffused light.

RESULTS OF TRANSMISSION MEASUREMENTS

Determinations of the transmission of Eastman 40 plates for red, green and blue light were made by the use of number 25, 58, and 49 Wratten filters,[5] respectively, placed over the opening of the lamp house. The receiver plate was in each case a Wratten and Wainwright Panchromatic.

It was also desired to obtain an idea of the average transmission of the plate for a band of wave length corresponding to the sensitivity curve of the plate. This band will hereafter be referred to as "actinic" light. The simplest means of doing this was found to consist in illumination with white light using another plate of the same type, namely an Eastman 40, as the receiver instead of a panchromatic plate.

Table 1 shows the results of a typical series of runs. The transmission is seen to change markedly with wave length, but shows no systematic variation with exposure time.

Because it was desired to obtain the transmission while the plate still retained the optical characteristics of an unexposed plate it was thought desirable to make a further test of the effect of exposure time. Several series of runs were made using various intensities and exposure times,

[5] These are the filters customarily used in three-color photography and their exact transmission characteristics can be found in "*Wratten Light Filters*," published by the Eastman Kodak Company, Rochester, N. Y.

covering a range sufficient to give developed densities varying from very small values up to about 3.66. No change in transmission was found over this range

TABLE 1. *Transmission by Eastman 40 Plates*

Red light

Exposure (seconds)	15	30	60	120	240	480
Transmission in %	42.5	43.7	44.7	41.6	43.5	43.0

Average 43.2%

Green light

Exposure (seconds)	15	30	60	120	240	480	960
Transmission in %	36.4	32.4	33.8	33.4	33.4	33.1	36.4

Average 34.1%

Blue light

Exposure (seconds)	30	60	120	240	480	1080	1920	3840
Transmission in %	10.2	8.1	8.5	9.3	9.6	9.3	9.1	8.5

Average 9.1%

Actinic band

Exposure (seconds)	7.5	15	30	60	120	240
Transmission in %	13.3	14.8	14.8	14.8	14.8	15.1

Average 14.6%

REFLECTION MEASUREMENTS

The method of measuring reflections is illustrated in Fig. 2, and consisted of a direct photographic comparison of the light reflected from the plate under investigation, with that reflected from a standard diffuse reflector under similar conditions.

Two exposures were necessary for each determination. In the first, light was allowed to fall normally upon a strip of the plate being tested which was placed at P, Fig. 2a. The light diffusely reflected at 45° was allowed to act on a narrow strip R of the receiver plate. Immediately following this, the plate, P, was replaced by a standard diffuse reflector as in Fig. 2b, and a second exposure was made just as before, with the difference that a strip of optical wedge, W, was now placed in contact with a new strip R' of the receiver plate. After development, spots of equal density on strips R and R' were picked out, and the wedge transmission for that point was measured. The reflection of the

standard reflector when multiplied by the wedge transmission was taken to be the reflection of the plate P.

If the plate P and the standard reflector were both perfectly diffusing reflectors, or if both plates gave the same distribution of reflected light, this assumption would be correct. The standard reflector was of the type

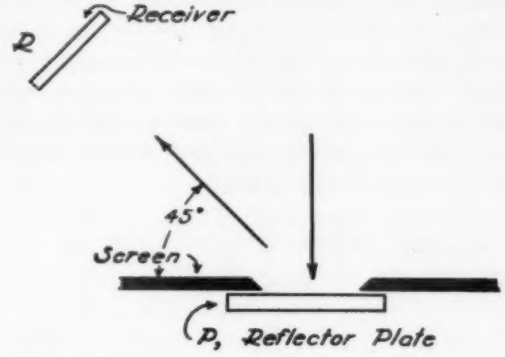


FIG. 2a

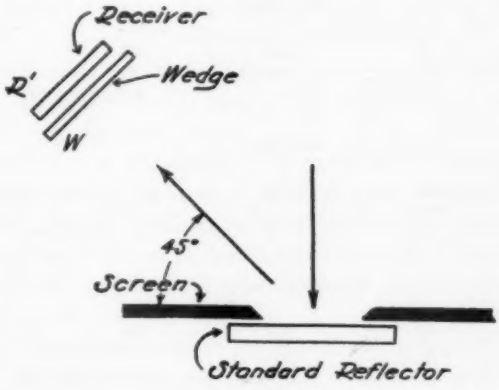


FIG. 2b.

well known in photometric work, namely a piece of depolished opal glass, which forms a fairly perfect diffuse reflector.[6] The shape of the distribution curve for Eastman 40 plates was not known, but rough visual observations showed the plate to be equally bright when viewed from various angles, while illuminated with normally incident light.

[6] International Critical Tables, 5, 261.

The width of both the reflector and receiver strips was 5/8 inch and the distance apart was 6½ inches. The length of strip in a direction perpendicular to the paper in Fig. 2 was about 4 inches. The duration of exposure was controlled by a stop watch and small knife switch, and the intensity was kept constant by a rheostat and milliammeter. The pair of associated successive exposures were made on adjacent portions of the same plate. The illumination over the surface of the receiver plate was not quite uniform, and the effect of this variation was allowed for by picking out spots on R and R' which were not only of equal density but which had geometrically similar locations on the two plates. Several screens and diaphragms (not shown) served to define the light beam and intercept stray light.

TABLE 2. *Reflection from Eastman 40 Plate*

		Red light		
Exposure (seconds)	60	120	240	480
Reflection in %	57.	55.	60.	60.
			Average 58.0%	
		Green light		
Exposure (seconds)	120	240	480	960
Reflection in %	55.	58.	58.	58.
			Average 57.3%	
		Blue light		
Exposure (seconds)	660	1200	2460	4800
Reflection in %	20.0	28.2	24.2	26.4
			Average 24.7%	
		Actinic band		
Exposure seconds)	60	120	240	480
Reflection in %	28.2	28.2	27.8	28.2
			Average 28.1%	

RESULTS OF REFLECTION MEASUREMENTS

The reflection of red, green, and blue light was determined, using Wratten filters numbers 25, 58, and 49, as used in the transmission experiments, with Wratten and Wainwright Panchromatic receiver plates. Reflection of "actinic" light was determined using white light and an Eastman 40 receiving plate.

Table 2 shows the results obtained. The reflection changes considerably with color, being a maximum for the longer wave lengths as would be expected from the yellowish color of the plates.

EFFECT OF EXPOSURE TIME ON REFLECTIONS

In these experiments only a small fraction of the light incident on the reflector plate reached the receiver, and it appeared quite possible that the large amount of energy which must fall on the reflector in order to get a reasonable density on the receiver, might change the reflection of the former. No change was observed with any of the exposure times used in obtaining the data of Table 2 but to make certain of this, a series of runs was made, measuring the reflection of plates which had been placed in the apparatus and exposed to the usual light intensity for widely varying periods. Measurements on plates which had been given pre-exposures of 180, 240, 480, 1080, 1920, 3840, and 7680 seconds to white light (0.158 candle power at 39 cm) were made and although these exposures were far in excess of those ordinarily given, there was found no change in reflection. The exposures did not produce a visible image prior to development.

DISCUSSION OF RESULTS

Having determined the reflection and transmission of a plate we can obviously calculate the fraction of the incident light which has been absorbed, by the relation that

reflection + absorption + transmission = incident light.

This gives the overall absorption of the plate without any attempt to differentiate between absorption by silver bromide, by gelatin or by the the glass plate. It seems probable, however, that nearly all the absorption must be due to the silver bromide in the visible region in which this work was done.

Table 3 shows the conclusions to be drawn regarding the disposal of the light which falls on the plate under conditions of normal photo-

TABLE 3. *Disposal of Light Falling on Plate*

Color of Light	Reflection	Transmission	Absorption
Red	58%	43%	−1%
Green	57%	34%	9%
Blue	25%	9%	66%
"Actinic"	28%	15%	57%

graphic exposure. It is seen that absorption follows roughly the sensitivity-wave-length curve of the material investigated, and not only is there a pronounced maximum found in the blue, but the plate proves to be fairly efficient for that color, some 66% of the incident light being absorbed.

High precision is not claimed for the results, but it is thought that they indicate to a fair degree of approximation the behavior of the plate under conditions of ordinary use. It would have been desirable to make reflection measurements at several other angles, but this could not be done without the construction of more complicated apparatus. It was hoped to extend the work to several other types of commercial plates besides the Eastman 40, but the time available proved insufficient.

The result of -1% for the absorption of red light may indicate something of the order of magnitude of the errors in this investigation, since the actual absorption was probably nearly zero, as the Eastman 40 plate failed to develop any noticeable density even after the longest of the exposures to red light.

SUMMARY OF RESULTS

1. The reflection and transmission of unexposed Eastman 40 plates have been determined for red, green, blue, and "actinic" light by a photographic method under conditions of normal use.

2. Reflection and transmission were found to change greatly with wave length, following the photographic sensitivity curve of the material.

3. It has been shown that reflection and transmission do not change during the course of exposures insufficient to produce visible darkening.

4. The plate was found to be a fairly efficient absorbing medium in the region of its greatest photographic sensitivity.

CALIBRATION OF STANDARD LAMPS FOR RELATIVE AND ABSOLUTE MEASUREMENTS

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[Physical Laboratory, University of Utrecht,
Utrecht, Holland. Received June 19, 1930]

Introduction

From intensity measurements made in our laboratory, it appeared to be necessary to have reliable and constant light sources available, in which the spectral energy distribution (relative standard lamp) as well as the absolute intensity at one particular wave length (absolute standard lamp) was known. Relative standard lamps are therefore necessary for the following purposes:

a. The photographic method for the comparison of spectral line intensities.

b. The measurement of the sensitivity curve of the combination monochromator-photocell, or monochromator-thermopile (or vacuum thermoelement).[1]

Absolute standard lamps are also used for the determination of the life-period of atoms which are brought into excited states in metallic vapors in flames.[2]

Standard Lamps for Relative Measurements

We give a short description of two quite different methods, by which the spectral energy distribution in a tungsten lamp in the visible region is measured. These methods are:

1. The photographic, which is quite independent of any temperature measurement.

2. The pyrometric, which depends on a temperature measurement with a Holborn-Kurlbaum pyrometer.

These two methods give an agreement of 1%–2% in the wave length region 7000A–4000A. Both methods have been worked out in

[1] One can easily see that a knowledge of the relative intensities can be of service. If a light source has the intensities I_1 and I_2 at the wave lengths λ_1 and λ_2, then due to the selective properties of the combination monochromator and photocell, the deflections μ_1 and μ_2 will be proportional to aI_1 and bI_2 where a and b are constants for the combination which must be determined. This may be done by making a measurement with a relative standard lamp in which the relative intensities I_3 and I_4 at λ_1 and λ_2 are known. The deflections μ_3 and μ_4 are now proportional to aI_3 and bI_4, so that $a/b=\mu_3/\mu_4\times I_4/I_4$ and therefore $I_1/I_2=\mu_1/\mu_2\times \mu_4/\mu_3\times I_3/I_4$.

[2] L. S. Ornstein and E. F. van der Held, Annalen der Physik, *85*, S. 953; 1928.

this laboratory by van Alphen[3] and have already been published. We shall only describe the principle of the two methods and refer to the above-mentioned publications for the measurements. Further, a new temperature measurement will be described, which depends on the fact that the temperature is a function of the energy ratio for two colors. The advantage of this method is that it is possible to carry out temperature measurements on lamps which cannot be measured pyrometrically.

THE PHOTOGRAPHIC METHOD FOR RELATIVE INTENSITY MEASUREMENTS

From the lamp P, which burns on constant voltage, we obtain an image at P_1 by means of the lens L_1, double monochromator[4] and L_2. This image is in one particular color, determined by the position of the middle slit S^2 of the monochromator. The energy of this image $E(\lambda)d\lambda$ is measured for the various wave lengths with a thermopile T and galvanometer. The thermopile used in gray over the visible region.

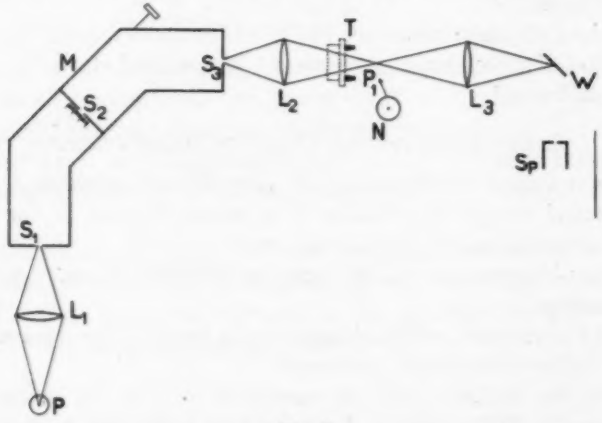

FIG. 1.

In order to find the relative intensity distribution as function of the wave length, we must divide the galvanometer deflection by the corresponding wave-length region, i.e., a region in which we assume that

[3] H. C. van Alphen, Dissertation, Utrecht; 1927, and Annalen der Physik, *85*, S 1058; 1928. The principle of the photographic method has been described by Ornstein, Scripta Universitatis Atque Bibliothecae Hierosolymitanarum; 1923.

[4] P. H. van Cittert, Zeitschrift für Instrumentenkunde, *46*, S. 557; 1926.

every wave length has the same intensity. The extent of this region is determined by an accurate measurement.

If we take out the middle slit of the monochromator, then we obtain at P_1 a white image of known spectral energy distribution. By means of this image an arbitrary continuous light source is calibrated photographically. An image of P_1 is thrown on a diffuse reflector W by the lens L_3, and the reflected light photographed with the spectrograph Sp. On the same plate, density marks are photographed by one of the known methods. The lamp which is to be calibrated is now put into the position P_1 and the spectrum of the lamp is photographed on the same plate with the same optical arrangement and the same exposure time. In this way the selective properties of the spectrograph are eliminated. The diffuse reflector is used in order to eliminate the effect of chromatic aberration in the lenses.

By this method we have measured the spectral energy distribution of a tungsten-band lamp. For the measurements on glowing tungsten spirals reference may be made to the publication of van Alphen.[5] It appears that the intensity distribution for a tungsten spiral and for a tungsten-band lamp can be described by the law of Wien.

THE PYROMETRIC METHOD

This method depends on the fact that the spectral energy distribution in the visible of glowing tungsten can be described by the color temperature T_c, that is to say, for every temperature of the tungsten, there corresponds a temperature T_c of a black body, at which the latter has practically the same energy distribution in the visible region as the tungsten at temperature T.

A. MEASUREMENTS ON BAND-LAMPS.

The black body temperature of a glowing tungsten band is measured with an optical pyrometer. The color temperature may be found from the black body temperature by means of a table given by Zwikker.[6]

B. MEASUREMENTS ON SPIRAL FILAMENT LAMPS.

If one examine an enlarged image of a glowing spiral, one sees immediately that the inner surface of the windings are brighter than the outer. The greater brightness of the inner surface is to be ascribed to reflection phenomena. We have considered the total energy of the middle windings as the sum of the emission of the darker parts and the emission of the brighter inner parts. The emission of the darker parts

[5] H. C. van Alphen, Annalen der Physik, *85*, S 1058; 1928.

[6] Zwikker, dissertation, Amsterdam; 1925.

can be described as a Wien function of the temperature T_e which corresponds to the black body temperature. In Fig. 2 curve 1 shows the emission of the darker parts. If the emission of the darker parts at 0.665μ is a, then that for the brighter parts is $qp_{0.665}\,a$, where q is the

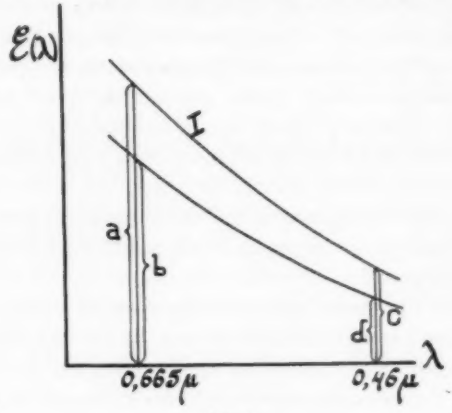

FIG. 2.

surface ratio and $p_{0.665}$ is the brightness ratio. $p_{0.665}$ can be measured with the optical pyrometer, as can the emission at 0.46μ. The total emission of the middle windings at 0.665μ and 0.46μ is the sum of the ordinates $a+b$ and $c+d$. We have now calculated the temperature T of a black body, so that

$$\frac{E_{0.665}T}{E_{0.46}T} = \frac{a+b}{c+d}.$$

By the photographic method it had already appeared that the spectral energy distribution of a spiral filament may be described by a color temperature. The color temperature found by the photographic method appeared to be the same as T found by means of the above mentioned equation.

TEMPERATURE MEASUREMENTS FROM MEASUREMENTS OF THE ENERGY IN TWO COLORS

This method depends on the fact that the temperature of a black body is a function of the energy ratio of two colors. This holds also for the color temperature, because the color temperature of a glowing body is the temperature of a black body which has the same spectral energy distribution.

It is known, that the spectral energy distribution of glowing tungsten in the visible can be described by the color temperature.

We have measured the black body temperature S of a tungsten band-lamp as function of the current. The observed temperature must be corrected for reflection of the glass bulb.

The red filter[7] of the Holborn-Kurlbaum pyrometer, Jena F 2745 has been examined by de Groot. He found an effective wave length of 0.653μ at a color temperature of 2000°K. One can then find the effective wave length at other color temperatures by the formula given by Foote.[8] Zwikker[9] gives the color temperature as function of the black body

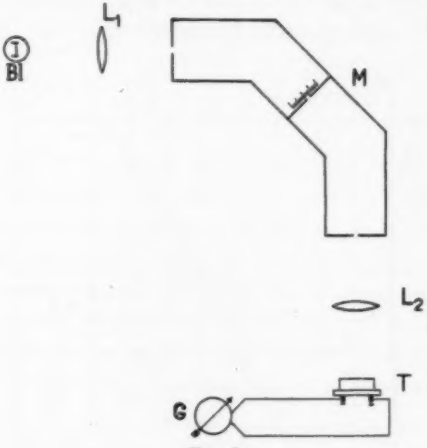

FIG. 3.

temperature at the wave length 0.665μ. In order to use this table we have calculated $S_{0.665\mu}$ from $_{0.653\mu}$ by means of the formula

$$1/T_c(1/\lambda_1+1/\lambda_2)=1/\lambda_1 S_{\lambda 1}-1/\lambda_2 S_{\lambda 2}.$$

After making the above mentioned corrections we find the color temperature of the tungsten band-lamp as function of the current.

TEMPERATURE MEASUREMENTS WITH THE MONOCHROMATOR

A tungsten band-lamp *B.L.* in which the relation between color temperature and current is known, is placed in front of a monochromator *M* and an image of the band is projected on a thermopile

[7] W, de Groot, Physica, *4*, 157; 1924.
[8] Zwikker, dissertation, Amsterdam; 1925.
[9] Zwikker, dissertation, Amsterdam; 1925.

T which is connected in series with a galvanometer G. The deflection for two colors red and green is measured, and the logarithm of the ratio plotted against $1/T_e$. If the wave-length region transmitted by the monochromator is small enough, the above relation can be described by a Wien curve. The spiral lamp to be calibrated is substituted for the band-lamp, the deflection for the appropriate colors is measured, and from the energy ratio, the color temperature is found on the calibration curve. We have compared a band-lamp and a spiral lamp both at the same T_e for the visible region. The T_e of the spiral lamp was found in the way described above. The agreement is within 1%. In Fig. 4, curve 1 gives the color-temperature of the middle windings; curve 2 the same quantity for the whole spiral. The ratio for the

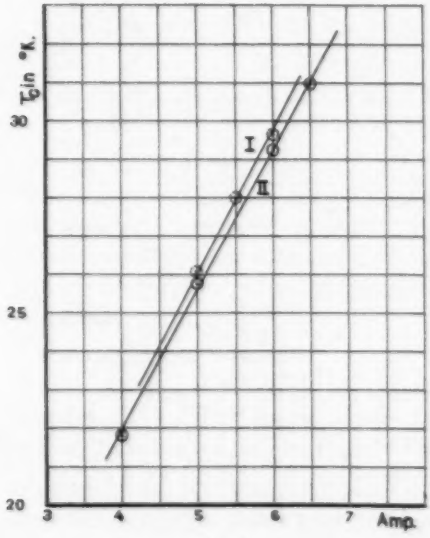

FIG. 4.

red and blue alters much more with color temperature, than that for red and green. The accuracy of a temperature measurement would be greater, if we could measure the ratio red-blue. The combination of thermopile and galvanometer is too insensitive to measure the energy in the blue with the lamps used, consequently a photocell is a more suitable instrument.

TEMPERATURE MEASUREMENTS WITH A PHOTO-CELL, RED AND BLUE FILTER

The arrangement[10] is shown in Fig. 5. The apparatus is surrounded by a metal box in order to shield it from disturbances. A cell filter with copper sulphate and quinine sulphate is placed in

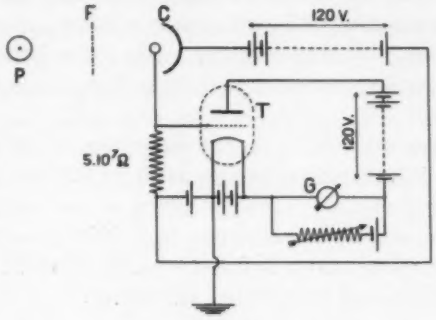

FIG. 5.

front of the photocell in order to absorb the far infrared and ultraviolet. The deflection is proportional to the light-intensity, as is shown by absorption measurements with different known concentrations of nigrosine. The arrangement was calibrated with a tungsten band-lamp in which the color temperature as function of the current had been measured with an optical pyrometer.

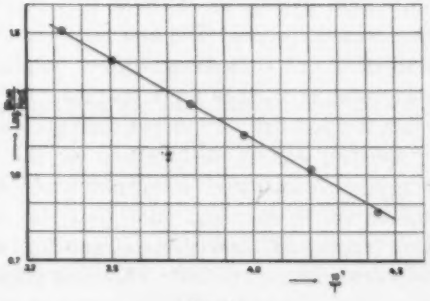

FIG. 6.

The logarithm of the ratio of the deflections for red and blue is plotted against $1/T_c$ (see Fig. 6).

[10] We are indebted to J. Custers of this laboratory for much information (hitherto unpublished) regarding the technique of these measurements. (Procceedings, Amsterdam; May, 1930).

The lamp to be calibrated is then substituted for the band-lamp and the color temperature found from the calibration curve.

By this method, one can determine the color temperature of a spiral lamp in a very simple way. This method has the advantage over the pyrometric method, that it is not necessary to carry out complicated calculations and measurements in order to determine the color temperature. With lamps provided with quartz windows, it appears that air leaks in, causing a layer of tungsten oxide to be formed on the glass bulb. With a pyrometer one cannot make temperature measurements with such lamps.

Since tungsten oxide is white, the absorption of the red and blue is equal, and one can therefore take the ratio for the two colors as a measure of the temperature. Through errors of measurements, the accuracy of a temperature measurement from 2000°K − 3000°K is ±0.5.° A method resembling that which has been used in this laboratory has been published recently by Clayton and Sharp.[11]

ABSOLUTE CALIBRATION OF A STANDARD LAMP

This calibration was done in two ways in order to have a check on the different measurements.

This was of great importance, since in this way as will appear the following methods and instruments were simultaneously tested. The absolute calibration of a thermopile according to the method of van Dyck[12] the emissivity of tungsten obtained by various investigation, the Holborn-Kurlbaum pyrometer used in this laboratory and the color temperature measurements. The first way was briefly as follows. The transmission of a cell containing copper sulphate and quinine sulphate was measured at various wave lengths. By means of a pyrometer, the color temperature at various currents was measured, and from this, the spectral energy distribution of the band-lamp was expressed in energy per A at 0.56μ. If one now multiplies these values by the transmission coefficients and adds them up, one obtains the energy of the lamp (expressed in energy per A at 0.56μ) which is received by a thermopile placed behind the cell. This sum appeared to be very little dependent on temperature, if the absorbing substance was chosen properly. The thermopile used was calibrated by the method of van Dyck. One measures then the energy per cm.^{2} which falls on the thermopile. Corresponding to the distance (corrected for the path in the

[11] Journal of the Optical Society of America, *20*, 62; 1930.
[12] Zeitschrift für Physikalische Chemie, Bd. *127*; 297.

cell) one measures the energy which is emitted by the lamp perpendicular to the band and transmitted by the cell. If one divides this value by the energy found as above (expressed in the unit there employed), one finds this unit, namely the energy per A at 0.56μ emitted by the lamp per solid angle perpendicular to the surface.

The second method is as follows:

From the measurement of the band-lamp temperature one calculates the true temperature. By means of the emissivity coefficients of tungsten (the values used were calculated from the measurements of Zwikker which are in good agreement with those found by Worthing)[13] it is possible to calculate which energies per A the lamp must emit for different currents at 0.56μ; these values must agree with those found by the first method.

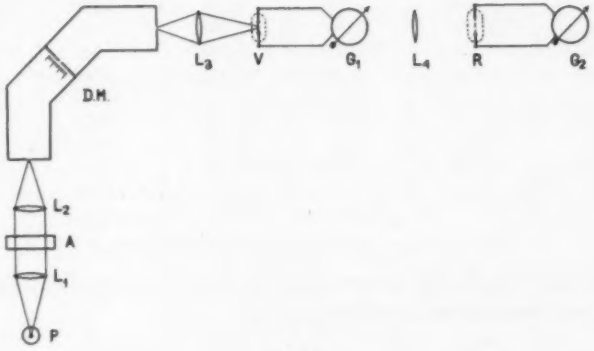

FIG. 7.

DETERMINATION OF THE COEFFICIENT OF TRANSMISSION

The absorption cell consists of two circular glass plates (6.5 cm diameter) between which a rubber ring (1.7 cm thick, inside and outside diameter 5.5, 6.5 cm respectively) is placed. Three screws fittings into two brass plates which are provided with circular openings (5.5 cm diameter) and which are larger than the glass plates, can be used to bring the glass plates into contact with the rubber ring. This apparatus is filled under the liquid which is to be used. In this way one obtains an absorption cell suitable for regular use. For the absorbing liquid, 200 g copper sulphate and 50 g quinine sulphate in 750 cc liquid were used. The light from

[13] Phys. Rev., *10*, 377; 1917.

a motor headlight lamp is rendered parallel by a lens L, and traverses the above described absorption cell. An image of the lamp is then projected on the slit of a double monochromator by means of a lens L_2. The monochromatic image of the last slit is in its turn projected on to the junction of a vacuum thermoelement[15] by means of a lens L_3. The resulting thermocurrent is measured with a thermorelay. (Galvanometer G_1, cylindrical lens L_4, thermorelay R and galvanometer G_2).

In Table 1 the measurements of the coefficients of transmission are given.

TABLE 1. *Transmission coefficients*

Wave length	Transmission	Wave length	Transmission	Wave length	Transmission
4223	0.019	4628	0.218	5243	0.384
4255	0.038	4675	0.246	5323	0.389
4290	0.066	4728	0.259	5410	0.388
4328	0.078	4780	0.304?	5500	0.378
4365	0.100	4838	0.298	5600	0.342
4405	0.119	4895	0.316	5700	0.295
4448	0.140	4960	0.328	5810	0.221
4490	0.156	5025	0.340	5925	0.147
4535	0.174	5093	0.359	6048	0.083
4580	0.193	5165	0.379	6178	0.040
				6325	0.014

TOTAL CALCULATED RELATIVE ENERGY

The next step is the determination of the black-body temperature for various currents with the pyrometer.

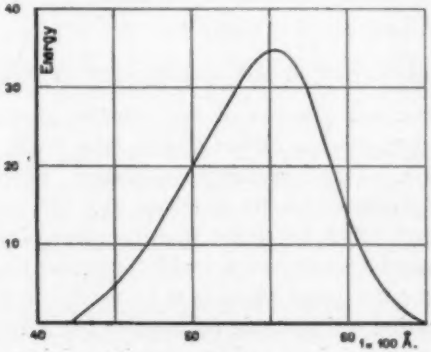

FIG. 8.

[15] Zeitschrift für Instrumentenkunde, *46*, S 557; 1926.
[15] Zeitschrift für Physik, *32*, S 575; 1925.

From this the true temperature and colour temperature can be calculated, and these are given in Table 2. From the color temperature it is easy to calculate the energy distribution (expressed in energy per A at 0.56μ). If these values are multiplied by the appropriate transmissions, then one obtains the results shown in Fig. 8 for 12 amp. The area of this figure represents the transmitted energy (in energy per A at 0.56μ). The different results are given in Table 2. One can see how small is the dependence on temperature of the area of these figures.

THE THERMOPILE

The surface thermopile which we used was compared with a thermopile which had been calibrated in absolute units, ($1\mu v \alpha 4.31 \times 10^{-4}$ watts per cm^2). The latter was enclosed in a heavy iron jacket, and the interspace filled with wool. In addition, the thermopile and jacket were both provided with a glass window and the whole instrument packed in wool. We have had no trouble from even a small creep of the zero. The galvanometer deflection was registered on photographic paper and measured under a travelling microscope. These values are given in Table 2.

TABLE 2.

Current Ampere	Temperature			Thermopile			Pyrometer
	Black	True	Color	Relative energy	Deflection	Energy per solid angle per A by 0.56μ	
10.04	2005	2165	2205	—	0.80[3]	*0.280$\times 10^{-4}$	0.268$\times 10^{-4}$
11.0	2115	2290	2337	—	—	—	0.508$\times 10^{-4}$
12.0	2215	2420	2473	353	0.75	0.875$\times 10^{-4}$	0.919$\times 10^{-4}$
13.0	2317	2550	2610	361	1.30	1.49 $\times 10^{-4}$	1.52 $\times 10^{-4}$
14.0	2421	2680	2750	368	2.09	2.34 $\times 10^{-4}$	2.39 $\times 10^{-4}$
15.0	2520	2805	2885	375	3.23[3]	3.56 $\times 10^{-4}$	3.69 $\times 10^{-4}$

From these values, together with the distance lamp-thermopile (corrected for the deviation of the rays in the absorption cell; corrected distance = 1 meter) it is possible to calculate the total energy radiated per unit angle by the lamp and transmitted by the cell. If one divides these values, as already mentioned above, by the relative energy values, then one finds the unit energy values, that is the energy radiated per solid angle per A at 0.56μ. These quantities are to be found in the penultimate column in Table 2 and in Fig. 9. We give here also a result which is

derived from a measurement with a cell of only half the thickness of the first cell. The experiments done with this cell give values which deviate by as much as 20% from those expected from the measurements with the pyrometer. From these deviations with different currents, it was concluded that infrared radiation must be transmitted by the filter, and indeed, a region of transmission with a maximum of 0.1% was found at 1.2μ. Since the maximum emission of the lamp occurs here, the disturbing influence on the measurements in the visible region was noticeable. Because of the necessity of a correction, the accuracy was of course smaller. One of these results (for the lowest current) is shown in Fig. 9 and in Table 2 with a $\times$ sign.

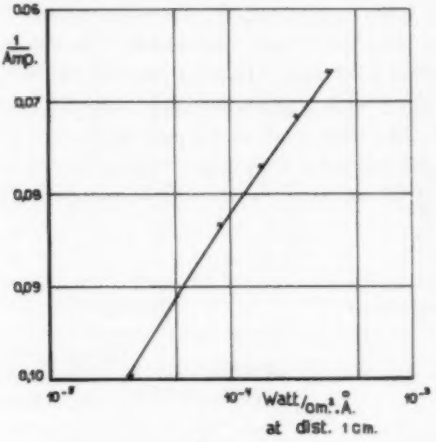

FIG. 9.

THE PYROMETER

According to the method already given, from the true temperature at different currents, the measured area of the flat filament lamp, and the known emission coefficient, the energies radiated per unit solid angle per A at 0.56μ were calculated (the last column of Table 2 and the full line in Fig. 9). The agreement of the values in the two columns is within 2% for the measurements with the absorption cell and within 6% for the narrower cell, which, considering how many steps are involved, may be considered satisfactory.

PARALLAX PANORAMAGRAMS FOR VIEWING BY REFLECTED LIGHT

By Herbert E. Ives

[463 West St., New York City. Received July 29, 1930]

The principle by which relief is attained in the parallax stereogram[1] and the parallax panoramagram[2] has heretofore been utilized only for the production of pictures for viewing by transmitted light. Theoretically the regular form of parallax stereogram and panoramagram should be capable of being viewed by reflected light, provided the photographic record of narrow strip images were printed on paper, or, if printed on glass or celluloid, provided that it be backed by a white surface to return light through the photographic layer. In practice, however, viewing by reflection is not practical, because the opaque line gratings used, particularly the panoramagram with its extremely narrow clear spaces, permit so little light to pass through them that the picture is quite inadequately illuminated. Even with the blackest attainable grating lines the light reflected back from the opaque portions will drown out the feeble relief image formed by reflection from the picture beneath.

In order to obtain sufficient illumination for working by reflected light, an entirely different structure from the opaque line grating is demanded. Such a structure is shown in section in Fig. 1. It consists of a transparent refracting medium provided on its front and back faces with opposed cylindrical ridges, of such curvatures that the front ridges focus parallel rays incident in any direction sharply upon

Fig. 1. *Section of structure for producing relief pictures for viewing by reflected light.*
a) *cylindrical ridges.*
b) *transparent base.*
c) *cylindrical ridges coated with photographic emulsion.*

the rear ridge surfaces, as shown by the dotted ray paths. If then the rear surface is coated with a diffusely reflecting material, such as

[1] F. E. Ives, U. S. Patent 725,567; 1903.
[2] C. W. Kanolt, U. S. Patent, 1,260,682; 1918.

white paint, any linear element of a rear ridge (perpendicular to the plane of the paper in the figure) will be illuminated only by light striking the front in a narrow sheet at a definite angle to the normal, and the diffuse reflection will be visible only at the same angle, since the rays simply retrace their paths (except for diffusion upward and downward). All of the incident light is returned (except for absorption losses), whereby the chief fault of the opaque line grating structure, if used for reflected light, is obviated.

Let us now suppose that the coating on the rear ridges is a photographic emulsion, and that it has impressed upon it the linear elements of a parallax panoramagram, leaving aside for the present the question of how to produce such a pictorial record. Suppose that this photographic record is backed by a white diffusing paint and that it is developed only to a low density, so that after the incident light passes twice through the picture the correct contrast is exhibited. We would then have a relief picture, visible by reflected light, adequately bright for satisfactory observation.

Before taking up the photographic problems involved in producing relief pictures of this sort, the character of the ridged structure shown in Figure 1 may be described in more detail. If the front cylindrical ridges are made with a radius of curvature s_1, and the medium has a refractive index n, simple lens theory gives for the thickness, assuming the rays outside the medium to be parallel, $f=s_1(n/n-1)$. The radius of curvature of the rear cylindrical ridges will be $f-s_1=s_1/n-1$, the front and rear ridges having the same axis. For a refractive index of 1.5, the thickness will be three times, and the rear cylinder radius twice, the front cylinder radius. The angle α between the axes of the extreme bundles of rays focussed on the rear cylinder determines the angle through which the relief picture will be visible, when the entire rear ridge is impressed with an appropriate panoramic view. For the case of a medium of refractive index 1.5, the angle α is 60°. These relations are shown in Figure 2.

The problem of producing a transparent ridged base of the sort shown in section in Fig. 1 (using celluloid as the material) and of coating it with a photographic emulsion, is one which has already been practically solved in another realm of photography.[3] The outstanding problem is the method of impressing the photographic strip images upon the back surface ridges. Contact printing from a flat negative produced

[3] The "Kodacolor" process.

in the usual way through a line grating is not feasible because of the extreme difficulty of securing exact registration of the linear structures of negative and print, especially if the latter is upon a material such as celluloid, susceptible of distortion.[4] Nor would this procedure, even if practicable, be desirable, for, as is discussed below, the ridged structure if properly utilized, makes possible relief pictures superior to any made upon flat surfaces. It is accordingly necessary to develop some scheme for forming the minute panoramic strips upon each ridge, and this scheme must, if it is to be practical, permit of the making of multiple copies.

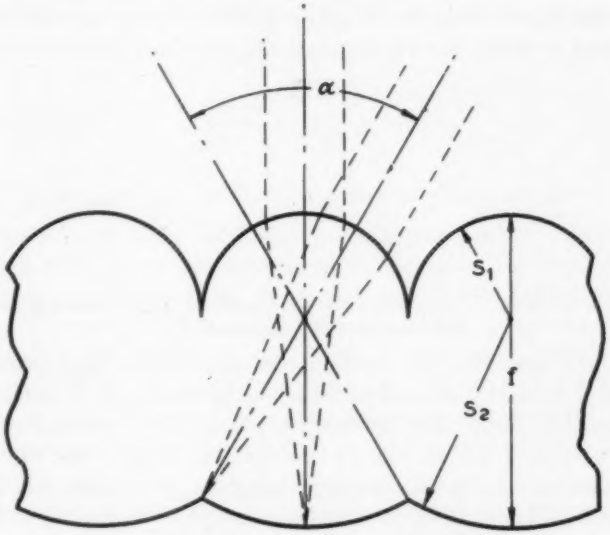

FIG. 2. *Exact form of ridged structure for relief pictures.*

I have experimented with two methods of making parallax pictures, applicable to ridged celluloid films. Both are projection printing methods, applicable as well to making parallax pictures embodying opaque line gratings and offering features of interest when so used. These methods will be described as worked out for making parallax

[4] I must record however that I have been shown a small picture of this general character, made by Mr. C. W. Kanolt, in which panoramic strips have been impressed by contact photographic printing on the rear, flat, surface, in satisfactory register with ridges on the front surface.

panoramagrams, showing relief through a wide angle, although the same procedure, simplified to give a pair instead of an infinity of points of view, will yield the simpler parallax stereogram.

The first method is characterized by the fact that the original negative from the object is made without the intermediation of any line grating or other structure. The principle invoked is easily grasped when it is pointed out that fundamentally what we require in a parallax panoramagram is an infinite series of pictures, each taken from a different angle. At first sight it might seem that to obtain and utilize such an infinity of views would be physically impractical. While of course a battery of closely juxtaposed cameras arranged in an arc about the object could be set up for making a large number of negatives; and a similar battery disposed for projecting superposed images

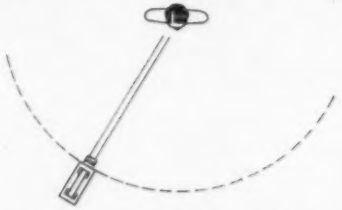

FIG. 3. *Method of making negative for relief picture by means of motion picture camera moved through arc around object.*

from these negatives for printing, the problems of registration encountered would be such as to make the procedure very cumbersome and unsatisfactory. But we have available, from another development in applied optics, namely the motion picture, just the apparatus necessary for producing large numbers of pictures which may be as little different from one another in point of view as we choose,—provided we have sufficient time at our disposal in our photography of the original object. The problem of producing a series of many pictures, each displaced from its neighbor in point of view by an infinitesimal angle, is solved—and this is the essential feature of this method—*by taking a motion picture film of the object with a motion picture camera which is given a slow continuous lateral movement during the exposure.* If for instance, as shown in Fig. 3, the camera is pivoted in the vertical axis through the object, and is moved through the entire semi-circular arc in front of the object in one minute, the strip of motion picture film exposed during this time will contain, at a normal taking rate, about 1500 separate views, an ample number to be con-

sidered "infinite" for our purpose. If a positive from this film is projected upon a screen in the normal manner from a projector fixed in position, it will exhibit the object as apparently slowly rotating about a vertical axis. If the projector is moved along an arc similar to that described by the camera, and the observer moves with the projector, he will see the object depicted as it would have been seen from the same relative positions of camera and object as those occupied by projector and screen.

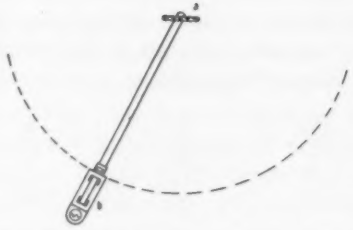

FIG. 4. *Method of printing parallex panoramagrams from motion picture film.*

In order to produce parallax panoramagrams from this motion picture film, it is clear that we have merely to substitute for the screen in the case just considered, an unexposed, emulsion coated ridged sheet of the sort shown in Fig. 1. Using the negative film, obtained by the procedure shown in Fig. 3, it is evident that after completion of the passage of the projector (Fig. 4-b) along the arc whose center is the ridged sheet (Fig. 4-a), a positive parallax panoramagram will be obtained upon completing the photographic development.

FIG. 5. *Alternative method for printing parallax panoramagrams from motion picture film.*

The apparatus for making parallax panoramagrams from the master motion picture film may be considerably simplified by virtue of the fact that relative rotation of the ridged sheet with respect to the projector is much easier to provide mechanically if the projector is kept stationary and the ridged sheet rotated. A schematic arrangement for achieving this is shown in Fig. 5 where the ridged sheet is mounted on a turntable rotated concurrently with the running through of the film in the stationary projector. (This arrangement could, of

course, be used also in taking the master film, provided the object photographed is small enough and otherwise adapted to be carried easily on a turntable.) This simplification is of very real value in making the prints from the film, since the film must be run through for each print, and any superfluous mechanical parts, subject to wear and tear, are to be avoided.

The second method of making parallax panoramagrams on a ridged base makes use of a large diameter lens, of the sort discussed in an earlier paper.[5] In this method, both the original negative and the prints are on ridged bases, and both are made by means of the large diameter lens. The complete procedure is illustrated in Fig. 6. In the upper diagram *a* is shown the process of making the negative *n*, from

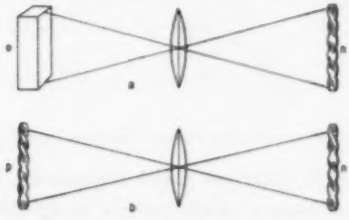

FIG. 6. *Method of making and printing parallax panoramagrams by means of large diameter lens.*

the original object *o*, by a single exposure to the large lens. Now this negative, while made up of minute panoramic strips, shows inverted or pseudoscopic instead of stereoscopic relief. If, however, this pseudoscopic negative is again photographed through the large lens, as shown in the lower diagram *b*, a positive picture is obtained, with the relief again inverted, that is, it is now correct.

An immediate difficulty is encountered if this procedure is followed exactly as outlined. The image of the ridged structure at *n* falling upon the second set of ridges at *p* causes interference or moiré patterns which will ruin the picture, and some means of avoiding this effect must be found. One way is to focus the image of *n* some distance in front of or behind *p*, so that the ridges at *n* are no longer clearly defined. If this is done, of course the relative position of the photographed object with respect to the plane of the picture is changed, but this need not be a serious objection if this displacement only amounts to a few millimeters, which is usually sufficient to avoid patterning. A

[5] J.O.S.A., *20*, 332; 1930.

second method is to use both taking and viewing ridges at an angle with the horizontal. Thus, if one set of ridges is at 45° and the other at 135°, their lines are mutually perpendicular and no patterning occurs, while the typical behavior of the ridges which is relied on to produce relief is preserved (upon choosing proper new curvatures of the ridges).

The second method of projection printing has the disadvantage, as compared with the first, of doubling a characteristic practical defect of parallax panoramagrams, namely, the poor depth of clear definition, due to the finite size and overlapping of the elementary linear elements of which the strip panoramas behind each grating space are composed. As against this disadvantage, however, is to be set the fact that the characteristic defect mentioned is greatly reduced by the use of the lenticular ridge structure, as compared with the opaque line grating,—with accurately formed ridges of sufficient size, the defect is theoretically absent. Consequently if the ridged structure is of a high degree of perfection, the image quality of the picture produced by the double use of the large lens, may still be expected to be satisfactory.

Before discussing the characteristics of the reflected light relief pictures, a few words may be said with regard to the use of these printing methods for parallax panoramagrams incorporating opaque line gratings, which are of necessity, as already explained, only feasible for use by transmitted light. A feature of the method using the motion picture film negative which is of practical value, is that it permits parallax panoramagrams to be made of any size or of any grating space from the one original, which is not the case where the first exposure is made, as is usual, through the grating. An advantage of both methods described, although probably only practically realizable with the motion picture film scheme, is that the panoramagram can be made in a light, non-fragile form, by substituting a celluloid sheet with grating lines on one side and photographic emulsion on the other for the heavy, double glass structure heretofore necessary. Even better, however, as a transparency is the relief picture obtained by using the double-ridged sheet described for use with reflected light, developed to proper density for viewing by transmitted light. As already mentioned, this has greater depth of clear definition than the opaque line grating form.

The form of relief picture for viewing by reflected light which has been described in the foregoing paragraphs, has been realized experi-

mentally, with sufficient success to establish the validity of the methods, by the use of celluloid sheet impressed with cylindrical ridges (50 to the inch) in an embossing press, and subsequently coated with photographic emulsion. After making the positive exposure (by either method) the print is developed to a low density, and the emulsion is backed with white or aluminum paint, and is ready for viewing. An extended light source is necessary, which must be placed above or below the plane of the eyes, in a position to avoid specular reflection. Lights to either side of the relief picture must be avoided as they are reflected by each ridge, causing a veil of light over the image. This limitation is of course undesirable but is generically similar to that encountered with all glossy or glass covered pictures. The experimental relief pictures, made in the manner described, while falling short of perfection because of defects in the ridged celluloid sheet available, are nevertheless quite striking. They offer, it is believed for the first time, pictures which may be handled and viewed like ordinary photographic prints on paper (at least when these are mounted on stiff card), but unlike them exhibit relief.

In conclusion the opportunity may be taken to stress particularly the extreme simplicity of the second method of making relief pictures—that in which both negative and printare made with a large diameter lens. Given the special camera, and the specially prepared photographic film the operations of making a picture in relief are, in so far as the negative is concerned, exactly those of ordinary photography, and, in so far as the positive is concerned, exactly those of projection printing. The print has the form of an ordinary photograph, and is viewed without the interposition of any instrument. It is difficult to imagine much greater simplification of stereoscopic photography.